On the Cycle-transitivity of the Dice Model

B. De Schuymer[1], **H. De Meyer**[1],
B. De Baets[2] and **S. Jenei**[3]

[1]Dept. Applied Mathematics and Computer Science
Ghent University, Krijgslaan 281, B-9000 Gent, Belgium
{Bart.DeSchuymer,Hans.DeMeyer}@rug.ac.be

[2]Dept. Applied Mathematics, Biometrics and Process Control
Ghent University, Coupure links 653, B-9000 Gent, Belgium
Bernard.DeBaets@rug.ac.be

[3]Inst. of Mathematics and Informatics
University of Pécs, Ifjúság u. 6, Hungary
jenei@ttk.pte.hu

Abstract

We introduce the notion of a dice model as a framework for describing a class of probabilistic relations. We investigate the transitivity of the probabilistic relation generated by a dice model and prove that it is a special type of cycle-transitivity that is situated between moderate stochastic transitivity or product-transitivity on the one side, and Łukasiewicz-transitivity on the other side. Finally, it is shown that any probabilistic relation with rational elements on a three-dimensional space of alternatives which possesses this particular type of cycle-transitivity, can be represented by a dice model. The same does not hold in higher dimensions.

Keywords: dice model, probabilistic relation, stochastic transitivity, T-transitivity, utility model.

1 Introduction

Two players play the following game. Player 1 erases the spots from the faces of three fair dice and writes one number from 1, 2, ..., 18 to each face. Each of them risks € 1, chooses one dice, they throw the dice, and the one having the bigger number on top of his dice wins the € 2. Since Player 1 puts the numbers to the dice it seems fair to let Player 2 choose his dice first. Of course, Player 2 tries to choose the best dice. Despite this, Player 1 wins in the long run.

Such an example for distributing the numbers over the three dice A, B, C is

$$A = \{1, 3, 4, 15, 16, 17\},\ B = \{2, 10, 11, 12, 13, 14\},\ C = \{5, 6, 7, 8, 9, 18\}.$$

Denoting by $P(X, Y)$ the probability that dice X wins from dice Y, we have $P(A, B) = 20/36$, $P(B, C) = 25/36$, $P(C, A) = 21/36$. We say that dice X is strictly better than dice Y (notation: $X \succ Y$) if $P(X, Y) > 1/2$, which reflects that dice X wins from dice Y in the long run. In the above example, it holds that $A \succ B$, $B \succ C$ and $C \succ A$, which means that for any dice A, B or C, one of the remaining dice is always strictly better. In this case, the relation 'better than' is not transitive and forms a cycle. The occurence of cycles has been observed in various psychological experiments related to gambling [16], to judgment of relative pitch in music [12] and to human preferences [14], for instance. Formulating the above observation in another way, if we interprete the probabilities $P(A, B)$, $P(B, C)$ and $P(C, A)$ as elements of a valued relation on the space of alternatives $\{A, B, C\}$, then this valued relation is even not weakly stochastic transitive.

The above example can be generalized in the following sense. Firstly, it is possible to consider an arbitrary (but fixed) number $m \geq 2$ of dice, each dice being characterized by a set A_i ($i = 1, 2, \ldots, m$) of numbers. Secondly, each set A_i may contain n_i numbers, with n_i not necessarily equal to six. In other words, we allow a dice to possess any number of faces, but do not care whether such a dice can be materialised and therefore maintain the dice terminology throughout this paper. Finally, we do not insist on having mutually distinct numbers on the faces of a single dice or among different dice.

Given a set of m generalized dice we will define the winning probabilities for each pair of dice and the set of dice will be called a dice model for the generated probabilistic relation. One of the main issues of the present paper is to investigate which kind of probabilistic relations can be generated by a dice model. To answer this question, we will first give a formal description of the dice model and then investigate the properties of the model which are conceptually related to transitivity. In particular, we will show that the dice model accounts for a specific type of transitivity, which we will call dice-transitivity.

A basic concept in the present study is that of a probabilistic relation, often also called a reciprocal or ipsodual relation. Probabilistic relations serve as a popular representation of various relational preference models [3, 8, 13].

Definition 1.1 *A probabilistic relation Q on a set of alternatives A is a mapping from A^2 to $[0, 1]$ such that for all a, b it holds that:*

$$Q(a, b) + Q(b, a) = 1\,. \tag{1}$$

If A is finite with cardinality m, then Q is called an m-dimensional probabilistic relation.

The number $Q(a, b)$ can, for instance, express the degree of preference of alternative a over alternative b. Probabilistic relations can be classified on the basis of their type of transitivity. Usually one considers as possible types of transitivity, various kinds of stochastic and fuzzy transitivity, but recently, also more general families of transitivity properties have been reported on [5, 6, 15].

On the other hand, various models for generating and representing probabilistic relations have already been established. Well known is the utility model in which to each alternative $x_i \in A$ a utility number $u_i \in \mathbb{R}$ is assigned and for which the generated probabilistic relations have strong transitivity properties [15]. At the other end of the transitivity scale, we encounter the probabilistic relations generated by the so-called multidimensional model [15], which possess the weak Łukasiewicz-transitivity property only.

We shall introduce a new probabilistic model (called dice model) to generate probabilistic relations. In this model, to each alternative $a_i \in A$ a multiset A_i consisting of n_i numbers is assigned. These multisets can be identified with the generalized dice discussed above. In contrast to Basile [1], our aim is not to resolve the cyclic behaviour commented on in the prologue, but to establish an appropriate type of transitivity accounting for it. We will show that this type of transitivity can be situated between that of the utility model and that of the multidimensional model.

2 The dice model

As stated before, the concept of a multiset is very well suited to formally describe a generalized dice. We recall that a multiset is a set with possibly repeated elements.

Definition 2.1 *Let V_n denote the class of multisets of cardinality n with strictly positive integer elements. Unless otherwise stated, the elements of a multiset $\{a_1, a_2, \ldots, a_n\} \in V_n$ are listed in non-descending order, i.e.: $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $V = \bigcup_{i=1}^{\infty} V_i$ denote the class of finite multisets with strictly positive integer elements. We will also use the notation $V_{n_1,n_2,\ldots,n_m} = V_{n_1} \times V_{n_2} \times \ldots \times V_{n_m}$ to denote the family of ordered collections $(M_1, M_2, \ldots, M_m)$ of m multisets $M_i \in V_{n_i}$. The multiset $M = \bigcup_{i=1}^{m} M_i$ is called the collective multiset of the given collection. Clearly, different collections may possess the same collective multiset.*

For our purposes, we will frequently make use of a special type of collection, called standard collection.

Definition 2.2 *A standard collection is a collection $(M_1, M_2, \ldots, M_m) \in V_{n_1,n_2,\ldots,n_m}$ for which the collective multiset M equals $\mathbb{N}[1, n_1 + n_2 + \cdots + n_m]$.*

$\mathbb{N}[a, b]$ denotes the set of integers in the interval $[a, b]$. Definition 2.2 implies that all elements of the collective multiset M are different and that every integer in M occurs once in just one of the composing multisets M_i. In fact, the multisets M_i of a standard collection are ordinary sets that constitute a partition of the ordinary set M.

We now indicate how we can unambiguously associate a probabilistic relation to a given collection of multisets.

Definition 2.3 *For any two multisets $A \in V_{n_1}$ and $B \in V_{n_2}$ we define:*

$$P(A, B) = \frac{1}{n_1 n_2} \left(\#\{(a, b) \in A \times B \mid a > b\}\right) , \tag{2}$$

$$I(A, B) = \frac{1}{n_1 n_2} \left(\#\{(a, b) \in A \times B \mid a = b\}\right) , \tag{3}$$

$$D(A, B) = P(A, B) + \frac{1}{2}\, I(A, B) . \tag{4}$$

The valued relation D on V defined by (4) clearly is a probabilistic relation.

It should be noted that, given a couple (A, B) of multisets, $P(A, B)$ (resp. $I(A, B)$) is the probability that an element drawn at random (with a uniform distribution) from the multiset A is strictly greater than (resp. equal to) an element drawn at random from the multiset B. If, for example, A is an ordinary integer set of cardinality n, then according to (2)–(4), we obtain $P(A, A) = (n - 1)/2n$ and $I(A, A) = 1/n$. In the context of fuzzy preference modelling [4], a strict preference relation P is assumed to be irreflexive ($P(A, A) = 0$) and an indifference relation to be reflexive ($I(A, A) = 1$). The valued relations introduced above, despite their probabilistic interpretation, do not fit into the framework of fuzzy preference structures. However, the probabilistic relation D can also be written as:

$$D(A, B) = P'(A, B) + \frac{1}{2} I'(A, B) \,,$$

where P' and I' are defined by:

$$\begin{aligned} P'(A, B) &= \max(P(A, B) - P(B, A), 0) \,, \\ I'(A, B) &= 1 - \mid P(A, B) - P(B, A) \mid \,. \end{aligned}$$

Now, P' (resp. I') can be interpreted as a strict preference (resp. indifference) relation. In particular, for an ordinary integer set A of any cardinality n, we obtain $P'(A, A) = 0$ and $I'(A, A) = 1$.

Definition 2.4 *A collection $(M_1, M_2, \ldots, M_m) \in V_{n_1,n_2,\ldots,n_m}$ is called a dice model for an m-dimensional probabilistic relation $Q = [q_{ij}]$, if it holds that*

$$q_{ij} = D(M_i, M_j) \,.$$

Q is called the probabilistic relation generated by the dice model.

A finite collection $(M_1, M_2, \ldots, M_m)$ of multisets, together with the associated m-dimensional probabilistic relation Q, can be represented by a weighted directed graph with m nodes. Node i corresponds to multiset M_i. Between every pair of nodes a directed arc is drawn and its direction is arbitrarily chosen. If an arc is drawn from node i to node j, then it carries the weight q_{ij}. It may be replaced by an arc from node j to node i carrying the weight $q_{ji} = 1 - q_{ij}$. Since $q_{ii} = 1/2$ for all i, for the sake of simplicity, loops at the graph nodes are not drawn. Figure 1 illustrates the graphical representation of the probabilistic relation generated by a dice model.

3 Standardization of a dice model

From the definitions introduced in the previous section it is clear that many different collections of m multisets can generate the same probabilistic relation. The question arises whether for a probabilistic relation generated by a collection of multisets, there always exists at least one standard collection that generates the same probabilistic relation. An affirmative answer to this question is obtained in this section.

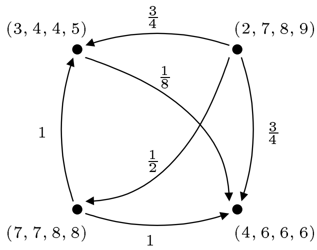

Figure 1: A 4-node graph representing the probabilistic relation generated by the multisets $\{3,4,4,5\}, \{2,7,8,9\}, \{4,6,6,6\}, \{7,7,8,8\}$.

Lemma 3.1 *Any collection* $C = (M_1, M_2, \ldots, M_m) \in V_{n_1,n_2,\ldots,n_m}$*, with collective multiset* M*, can be transformed into a collection* $C' = (M_1', M_2', \ldots, M_m') \in V_{n_1,n_2,\ldots,n_m}$*, with collective multiset* M'*, so that:*

$$\begin{array}{l} 1.\ C \text{ and } C' \text{ generate the same probabilistic relation};\\ 2.\ 1 \in M';\\ 3.\ v \in M' \Rightarrow v \in \mathbb{N}[1, n_1 + n_2 + \ldots + n_m];\\ 4.\ v \text{ occurs } n \text{ times in } M' \Rightarrow M' \cap \mathbb{N}[v+1, v+n-1] = \emptyset\,. \end{array} \tag{5}$$

Proof: The proposed transformation of C into C' is essentially the unique order-preserving renumbering of the elements of M satisfying conditions 2, 3 and 4. ∎

Example 3.1: To illustrate this transformation, let us consider the following example. The collection (M_1, M_2, M_3) with

$$\begin{cases} M_1 = \{2, 2, 11, 14, 15\} \\ M_2 = \{2, 3, 3, 5, 12\} \\ M_3 = \{8, 8, 8, 9, 10\} \end{cases}$$

is transformed into the collection (M_1', M_2', M_3') with

$$\begin{cases} M_1' = \{1, 1, 12, 14, 15\} \\ M_2' = \{1, 4, 4, 6, 13\} \\ M_3' = \{7, 7, 7, 10, 11\} \end{cases}$$

One can easily verify that $D(M_1, M_2) = D(M_1', M_2') = 3/5$, $D(M_2, M_3) = D(M_2', M_3') = 1/5$ and $D(M_3, M_1) = D(M_3', M_1') = 2/5$.

Theorem 3.1 *Any collection* $C = (M_1, M_2, \ldots, M_m) \in V_{n_1,n_2,\ldots,n_m}$ *can be transformed into a standard collection* $\tilde{C} = (\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_m) \in V_{2n_1,2n_2,\ldots,2n_m}$ *that generates the same probabilistic relation.*

Proof: We first transform C into C', using Lemma 3.1. Next, we will transform the collective multiset M' of the collection C' into a multiset $\tilde{M}$ corresponding to a standard collection $\tilde{C} \in V_{2n_1,2n_2,\ldots,2n_m}$ that generates the same probabilistic relation.

For each distinct number ℓ in the multisets of C' (each distinct number in M') we do the following. If ℓ occurs only once in M', then we replace it by $2\ell - 1$ and 2ℓ. So, the multiset $\tilde{M}_i$ of $\tilde{C}$ that corresponds to the multiset M'_i of C' containing ℓ, contains $2\ell - 1$ and 2ℓ instead. If the number ℓ occurs twice in M', we replace one ℓ by $2\ell - 1$ and $2\ell + 2$ and the other ℓ by 2ℓ and $2\ell + 1$. Generally speaking, if ℓ occurs t times in M', then we replace the j^{th} ℓ by $2\ell + j - 2$ and $2\ell + 2t - j - 1$. Note that the t equal numbers ℓ are arbitrarily ordered, each ordering possibly giving rise to a different standard collection.

We will now prove that C' and $\tilde{C}$ generate the same probabilistic relation. As a first step we note that, thanks to the fourth property in (5), for any two distinct numbers $a > b$ from M' that are respectively transformed into the pairs of numbers a_1, a_2 and b_1, b_2 contained in $\tilde{M}$, it holds that both a_1 and a_2 are strictly greater than b_1 and b_2. Therefore the contribution to $D(M'_i, M'_j)$ originating from different numbers in M'_i and M'_j (the $P(M'_i, M'_j)$ part) equals the contribution to $D(\tilde{M}_i, \tilde{M}_j)$ originating from the transformed pairs of those numbers in $\tilde{M}_i$ and $\tilde{M}_j$. It remains to investigate whether the $I(M'_i, M'_j)$ contribution to $D(M'_i, M'_j)$ is reproduced by the transformed numbers (which are mutually different). To that aim let us consider the case where ℓ occurs in at most 2 multisets, say M'_i and M'_j, k times in M'_i and $t - k$ times in M'_j with $k \in \{0, 1, \ldots, t\}$ and $t > 1$. Without loss of generality we can assume that no other number but ℓ occurs in M'_i and M'_j and that $\ell = 1$.

According to the proposed transformation, $\tilde{M}_i$ contains the $2k$ numbers j_1, $2t - j_1 + 1$, j_2, $2t - j_2 + 1$, $\ldots$, j_k, $2t - j_k + 1$, with $1 \leq j_1 < j_2 < \cdots < j_k \leq t$, whereas $\tilde{M}_j$ contains the remaining numbers in $\mathbb{N}[1, 2t]$. Counting the number s of couples $(a, b) \in \tilde{M}_i \times \tilde{M}_j$ for which $a > b$, we obtain in increasing order of a:

$$
\begin{aligned}
s = {} & (j_1 - 1) + (j_2 - 2) + \cdots + (j_k - k) + \\
& (2t - j_k - k) + (2t - j_{k-1} - k - 1) + \cdots + (2t - j_1 - 2k + 1) \\
= {} & 2k(t - k)\,.
\end{aligned}
$$

Hence, $D(\tilde{M}_i, \tilde{M}_j) = 2k(t-k)/(4k(t-k)) = 1/2$ which is equal to $D(M'_i, M'_j)$. Finally, the generalization to the case where the same number ℓ occurs in three or more multisets is straightforward. ∎

Example: Continuing the same example as before, the collection (M_1', M_2', M_3') can for instance be transformed into the collection $(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ with

$$\begin{cases} \tilde{M}_1 = \{2, 3, 4, 5, 23, 24, 27, 28, 29, 30\} \\ \tilde{M}_2 = \{1, 6, 7, 8, 9, 10, 11, 12, 25, 26\} \\ \tilde{M}_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22\} \end{cases}$$

One can easily verify that $D(M_1', M_2') = D(\tilde{M}_1, \tilde{M}_2) = 3/5$, $D(M_2', M_3') = D(\tilde{M}_2, \tilde{M}_3) = 1/5$ and $D(M_3', M_1') = D(\tilde{M}_3, \tilde{M}_1) = 2/5$. ∎

Theorem 3.1 enables us to focus without loss of generality solely upon standard collections when investigating the transitivity properties of the probabilistic relations generated by collections of multisets. We will call standard collections $\in V^3$ (resp. $\in V^4$) standard triplets (resp. standard quartets). Note that for standard collections it holds that $D(M_i, M_j) = P(M_i, M_j)$ for all $i \neq j$.

4 Cycle-transitivity

Usually, in the context of probabilistic relations, two types of transitivity are investigated: stochastic transitivity [8, 11] and T-transitivity [9], with T a triangular norm. A t-norm T is an increasing, commutative and associative binary operation on $[0, 1]$ with neutral element 1 [10].

Definition 4.1 *Let g be an increasing $[1/2, 1]^2 \to [0, 1]$ mapping. A probabilistic relation Q on A is called g-stochastic transitive if for any $(a, b, c) \in A^3$ it holds that*

$$(Q(a, b) \geq 1/2 \wedge Q(b, c) \geq 1/2) \ \Rightarrow\ Q(a, c) \geq g(Q(a, b), Q(b, c)) \,. \tag{6}$$

This definition includes many well-known types of stochastic transitivity. Indeed, g-stochastic transitivity is known as strong stochastic transitivity when $g = \max$, moderate stochastic transitivity when $g = \min$, weak stochastic transitivity when $g = 1/2$, λ-transitivity, with $\lambda \in [0, 1]$, when $g = \lambda \max + (1 - \lambda) \min$ [2, 11].

It is clear that strong stochastic transitivity implies λ-transitivity, which implies moderate stochastic transitivity, which, in turn, implies weak stochastic transitivity.

Definition 4.2 *A valued relation R on A is called T-transitive w.r.t. a t-norm T if for any $(a, b, c) \in A^3$ it holds that:*

$$T\left(R(a, b), R(b, c)\right) \leq R(a, c) \,. \tag{7}$$

Prototypical examples of continuous t-norms are the minimum operator $T_{\mathbf{M}}$ ($T_{\mathbf{M}}(x,y) = \min(x,y)$), the algebraic product $T_{\mathbf{P}}$ ($T_{\mathbf{P}}(x,y) = x\,y$) and the Łukasiewicz t-norm $T_{\mathbf{L}}$ ($T_{\mathbf{L}}(x,y) = \max(x+y-1,\ 0)$). Clearly, $T_{\mathbf{M}}$-transitivity (also called min-transitivity) implies $T_{\mathbf{P}}$-transitivity (also called product-transitivity) and the latter implies $T_{\mathbf{L}}$-transitivity (also called Łukasiewicz-transitivity).

It has been shown by the present authors that for probabilistic relations stochastic transitivity and T-transitivity are only very special cases of a more general type of transitivity, called cycle-transitivity [6]. For a probabilistic relation Q on A, we write $q_{ab} := Q(a,b)$. In the framework of cycle-transitivity, the quantities $\alpha_{abc} = \min(q_{ab}, q_{bc}, q_{ca})$, $\beta_{abc} = \text{median}(q_{ab}, q_{bc}, q_{ca})$ and $\gamma_{abc} = \max(q_{ab}, q_{bc}, q_{ca})$ are defined for all (a,b,c). Obviously, $\alpha_{abc} \leq \beta_{abc} \leq \gamma_{abc}$. Also, the notation $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\}$ is used. Remark that any three nodes can always be labelled a, b, c such that either

$$\alpha_{abc} = q_{ab}, \quad \beta_{abc} = q_{bc}, \quad \gamma_{abc} = q_{ca}\,, \tag{8}$$

or

$$\alpha_{abc} = q_{ab}, \quad \beta_{abc} = q_{ca}, \quad \gamma_{abc} = q_{bc}\,. \tag{9}$$

Definition 4.3 [6] *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;

(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geq 1\,.$$

The class of upper bound functions is denoted $\mathcal{U}$.

The function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha,\beta,\gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha) \tag{10}$$

is called the *dual lower bound function* of a given upper bound function U.

Definition 4.4 [6] *A probabilistic relation Q on A is called cycle-transitive w.r.t. an upper bound function U, if for any $(a,b,c) \in A^3$ it holds that:*

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})\,, \tag{11}$$

where L is the dual lower bound function of U.

Due to the built-in duality, it holds that if (11) is true for some (a, b, c), then it is also true for any permutation of (a, b, c). Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any (a, b, c) that are not cyclic permutations of one another, e.g. (a, b, c) and (c, b, a).

This definition implies that if a probabilistic relation Q is cycle-transitive w.r.t. U_1 and $U_1(a, b, c) \leq U_2(a, b, c)$ for all $(a, b, c) \in \Delta$, then Q is cycle-transitive w.r.t. U_2.

In particular, $T_{\mathbf{M}}$-transitivity corresponds to $U = U_{\mathbf{M}}$ with $U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta$ and is equivalent to $\alpha_{abc} + \gamma_{abc} = 1$. $T_{\mathbf{P}}$-transitivity corresponds to $U = U_{\mathbf{P}}$ with $U_{\mathbf{P}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta$, and $T_{\mathbf{L}}$-transitivity corresponds to $U = U_{\mathbf{L}}$ with $U_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$. Furthermore, strong (moderate, weak) stochastic transitivity corresponds to $U = U_{ss}$ ($U = U_{ms}$, $U = U_{ws}$) with $U_{ss}(\alpha, \beta, \gamma) = \beta$, $U_{ms}(\alpha, \beta, \gamma) = \gamma$ and $U_{ws}(\alpha, \beta, \gamma) = \beta + \gamma - 1/2$ when $\beta \geq 1/2$ and $\alpha \neq 1/2$, $U_{ss}(\alpha, \beta, \gamma) = U_{ms}(\alpha, \beta, \gamma) = U_{ws}(\alpha, \beta, \gamma) = 2$ when $\beta < 1/2$ and $U_{ss}(\alpha, \beta, \gamma) = U_{ms}(\alpha, \beta, \gamma) = U_{ws}(\alpha, \beta, \gamma) = 1/2$ when $\alpha = 1/2$ [5, 6]. It follows that $T_{\mathbf{M}}$-transitivity implies strong stochastic transitivity and that moderate stochastic transitivity implies $T_{\mathbf{L}}$-transitivity. Yet another type of cycle-transitivity is that shown by the utility model [15], for which $U = U_u$ with $U_u(\alpha, \beta, \gamma) = \max(\beta, 1/2)$.

5 Transitivity of the dice model

From here on we will use (i, j, k) instead of (a, b, c) as specifiers of a cycle, since we only consider a finite, and therefore countable, number of dice. In this section, we will show by means of the concept of cycle-transitivity that the type of transitivity exhibited by a probabilistic relation Q generated by a dice model, can be situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity. This is expressed in the next four theorems.

Theorem 5.1 *Not all probabilistic relations generated by a dice model are $T_{\mathbf{P}}$-transitive.*

Proof: The probabilistic relation generated by Example 3.1 is not $T_{\mathbf{P}}$-transitive since it holds that $D(M_2, M_1) = 2/5$, $D(M_1, M_3) = 3/5$, $D(M_2, M_3) = 1/5$ and $2/5 \cdot 3/5 > 1/5$. ∎

Since $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity, clearly not all probabilistic relations generated by a dice model are $T_{\mathbf{M}}$-transitive.

Theorem 5.2 *Every probabilistic relation generated by a dice model is $T_{\mathbf{L}}$-transitive.*

Proof: Firstly, we note that in view of Theorem 3.1 the proof must only be given for an arbitrary standard collection $(M_1, M_2, \ldots, M_m) \in V_{n_1,n_2,\ldots,n_m}$. Furthermore, we only need to show that the elements of the generated probabilistic relation $Q = [q_{ij}]$ satify the double inequality

$$0 \leq \alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 \leq 1$$

for all $i < j < k$. Let us define

$$x_{ijk} = \frac{1}{n_i n_j n_k} \# \{(x_i, x_j, x_k) \in M_i \times M_j \times M_k \mid x_i > x_j > x_k\} \,,$$

then, since the collection is standard, it follows that

$$x_{ijk} + x_{ikj} + x_{jik} + x_{jki} + x_{kij} + x_{kji} = 1 \,.$$

On the other hand, it holds that $q_{ij} = x_{ijk} + x_{ikj} + x_{kij}$, $q_{jk} = x_{ijk} + x_{jik} + x_{jki}$, and $q_{ki} = x_{kij} + x_{kji} + x_{jki}$. Consequently,

$$\alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 = q_{ij} + q_{jk} + q_{ki} - 1 = x_{ijk} + x_{jki} + x_{kij} \,,$$

and the value of the last expression always lies in $[0, 1]$, which completes the proof. ∎

The reverse statement is not always true, as is illustrated by:

Theorem 5.3 *Not all $T_{\mathbf{L}}$-transitive probabilistic relations can be generated by a dice model.*

Proof: We will indicate a family of three-dimensional $T_{\mathbf{L}}$-transitive probabilistic relations that cannot be generated by a triplet of multisets. Indeed, let us consider the three-dimensional probabilistic relation Q with rational elements satisfying $q_{12} \neq 1$, $q_{23} \neq 1$, $q_{31} \neq 1$ and $q_{12} + q_{23} + q_{31} = 2$. Clearly such relations exist and are $T_{\mathbf{L}}$-transitive. Suppose that there exists a standard triplet (M_1, M_2, M_3) with $M_1 \in V_{n_1}$, $M_2 \in V_{n_2}$ and $M_3 \in V_{n_3}$, such that

$$q_{12} + q_{23} + q_{31} = 2 \,.$$

Let us first consider the case where the largest number $n = n_1+n_2+n_3$ is not in a multiset of cardinality one. Without loss of generality we can assume that M_1 contains the number n. Let us consider the standard triplet (M_1', M_2', M_3'), with corresponding elements q'_{12}, q'_{23} and q'_{31} of the probabilistic relation Q', that is obtained from (M_1, M_2, M_3) after removing n from M_1. Hence $M_1' \in V_{n_1-1}$ and we have in particular:

$$\begin{aligned}
q'_{12} &= \frac{n_1 n_2 q_{12} - n_2}{(n_1-1)n_2} = \frac{n_1 q_{12} - 1}{n_1 - 1} = q_{12} + \frac{q_{12}-1}{n_1-1}\,,\\
q'_{23} &= q_{23}\,,\\
q'_{31} &= \frac{n_3 n_1 q_{31}}{n_3(n_1-1)} = \frac{n_1 q_{31}}{n_1-1} = q_{31} + \frac{q_{31}}{n_1-1}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
q'_{12} + q'_{23} + q'_{31} &= q_{12} + q_{23} + q_{31} + \frac{q_{12}+q_{31}-1}{n_1-1}\\
&= 2 + \frac{1-q_{23}}{n_1-1} > 2\,,
\end{aligned}$$

which is a contradiction since by Theorem 5.2 the above sum should not exceed 2 for a standard triplet. Therefore, (M_1, M_2, M_3) is not a standard triplet.

There remains the case of a standard triplet with n contained in a multiset of cardinality 1. Suppose $n_1 = 1$ and $M_1 = \{n\}$ with $n = 1 + n_2 + n_3$. It follows that $q_{31} = 0$ and $q_{12} = 1$, but the latter equality is clearly not in agreement with the basic assumptions. Finally, since Q cannot be generated by a standard triplet, due to Theorem 3.1, it cannot be generated by an arbitrary triplet. This completes the proof. ∎

In the case of $T_{\mathbf{P}}$-transitive probabilistic relations, we can give conditions under which their generation by means of a dice model is always possible.

Theorem 5.4 *Every three-dimensional $T_{\mathbf{P}}$-transitive probabilistic relation Q with rational elements can be generated by a dice model.*

Proof: Consider a three-dimensional probabilistic relation Q with rational elements q_{ij}. According to the discussion preceding Definition 4.3, we assume that the elements of Q can be relabelled such that (8) holds, or

$$q_{12} = \alpha_{123},\ q_{23} = \beta_{123},\ q_{31} = \gamma_{123}.$$

The proof for the case where (9) holds is similar and will not be given explicitly here. Since $\alpha_{123},\ \beta_{123},\ \gamma_{123}$ are rational numbers, they have a least common denominator which we will call n. Furthermore, let $p = n\,\alpha_{123}$, $q = n\,\beta_{123}$ and $r = n\,\gamma_{123}$. In this notation, $T_{\mathbf{P}}$-transitivity means that the double inequality

$$qr \le n(p+q+r-n) \le n(p+q) - pq$$

holds.

Since $qr \le nq \le n(p+q) - pq$, we can distinguish two cases for the construction of the standard triplet. The first case is the one where $p,\ q,\ r$ satisfy:

$$qr \le n(p+q+r-n) \le nq\,,$$

or, equivalently:

$$(n-q)(n-r) \le np \le n(n-r)\,. \tag{12}$$

Then we define:

$$\begin{aligned} M_1 &= \mathbb{N}[1,r] \cup E\,,\\ M_2 &= \mathbb{N}[r+1,n-q+r] \cup E^c\,,\\ M_3 &= \mathbb{N}[n-q+r+1,2n-q+r]\,, \end{aligned} \tag{13}$$

with E an $(n-r)$-dimensional subset of $\mathbb{N}[2n-q+r+1,3n]$ and $E^c = \mathbb{N}[2n-q+r+1,3n] \setminus E$. E^c is q-dimensional. From (13) it is immediately clear that $D(M_2,M_3) = nq/n^2 = \beta_{123}$ and $D(M_3,M_1) = nr/n^2 = \gamma_{123}$. Depending upon the choice of E we obtain that $D(M_1,M_2)$ can vary in steps of $1/n^2$ from $(n-q)(n-r)/n^2$ when $E = \mathbb{N}[2n-q+r+1,3n-q]$ to $n(n-r)/n^2$ when $E = \mathbb{N}[2n+r+1,3n]$. In particular, for all p satisfying (12) at least one subset E can be found for which $D(M_1,M_2) = np/n^2 = \alpha_{123}$.

The second case is the one where $p,\ q,\ r$ satisfy:

$$nq \le n(p+q+r-n) \le n(p+q) - pq\,,$$

or, equivalently:

$$n(n-p) \le nr \le n^2 - pq\,. \tag{14}$$

We define:

$$\begin{aligned} M_1 &= \mathbb{N}[1,n-p] \cup E^c\,,\\ M_2 &= \mathbb{N}[n-p+q+1,2n-p+q]\\ M_3 &= \mathbb{N}[n-p+1,n-p+q] \cup E\,, \end{aligned} \tag{15}$$

where now E is an $(n-q)$-dimensional subset of $\mathbb{N}[2n-p+q+1,3n]$ and $E^c = \mathbb{N}[2n-p+q+1,3n] \setminus E$ is p-dimensional. From (15) it is immediately clear that $D(M_1,M_2) = np/n^2 = \alpha_{123}$ and $D(M_2,M_3) = nq/n^2 = \beta_{123}$.

Depending upon the choice of E we obtain that $D(M_3, M_1)$ can vary from $n(n-p)/n^2$ when $E = \mathbb{N}[2n-p+q+1, 3n-p]$ to $(n(n-p)+(n-q)p)/n^2 = n^2 - pq$ when $E = \mathbb{N}[2n+q+1, 3n]$. Hence for all r satisfying (14) at least one subset E can be found for which $D(M_3, M_1) = nr/n^2 = \gamma_{123}$. ∎

Since $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity, the construction in Theorem 5.4 can be used to establish a standard triplet that generates a given three-dimensional $T_{\mathbf{M}}$-transitive probabilistic relation with rational elements. However, this construction can be simplified in the following way, taking into account that with the same notations as before, $T_{\mathbf{M}}$-transitivity means that $p + r = n$. For the case that (8) holds we can choose

$$\begin{aligned} M_1 &= \mathbb{N}[1, r] \cup \mathbb{N}[r+2n+1, 3n]\,, \\ M_2 &= \mathbb{N}[r+1, r+n-q] \cup \mathbb{N}[r+2n-q+1, r+2n]\,, \\ M_3 &= \mathbb{N}[r+n-q+1, r+2n-q]\,, \end{aligned}$$

which immediately leads to $q_{12} = D(M_1, M_2) = \alpha_{123}$, $q_{23} = D(M_2, M_3) = \beta_{123}$ and $q_{31} = D(M_3, M_1) = \gamma_{123}$.

We now want to characterize more precisely the transitivity of the probabilistic relations generated by a dice model and therefore have reached the point where the main results of this paper can be formulated.

Theorem 5.5 *Every probabilistic relation generated by a dice model is cycle-transitive w.r.t. the upper bound function U_D defined by:*

$$U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma\,. \tag{16}$$

Cycle-transitivity w.r.t. the upper bound function U_D will be called dice-transitivity.

Proof: We are not able to formulate a direct proof in the style of the one of Theorem 5.2. Instead, we will establish a proof by induction. In view of Theorem 3.1, we can restrict the proof to probabilistic relations Q generated by a standard triplet. Furthermore, we only give the proof for the case where the 123-loop is of type (8). The proof for a 123-loop of type (9) is completely similar and is left to the reader. In the present case we have that $q_{12} = D(M_1, M_2) = \alpha_{123}$, $q_{23} = D(M_2, M_3) = \beta_{123}$ and $q_{31} = D(M_3, M_1) = \gamma_{123}$. For the sake of simplicity, we drop the subscript 123 from here onwards.

By induction, suppose there exists a standard triplet (M_1, M_2, M_3) with $M_1 \in V_{n_1}$, $M_2 \in V_{n_2}$, $M_3 \in V_{n_3}$ that generates a dice-transitive probabilistic relation, i.e.

$$\alpha\beta \leq \alpha + \beta + \gamma - 1 \leq \beta + \gamma - \beta\gamma\,, \tag{17}$$

where the lower bound is found by $1-U_D(1-\gamma, 1-\beta, 1-\alpha) = \alpha\beta$. The largest number occurring in one of the multisets of the triplet equals $n_1 + n_2 + n_3$.

We now construct a new standard triplet by attributing to one of the three multisets the additional number $n = n_1 + n_2 + n_3 + 1$. We therefore have to distinguish three cases. We systematically use accents to denote quantities related to the newly constructed standard triplet.

Case 1: M_1 is attributed the additional number n. We obtain:

$$\begin{aligned}
n_1' &= n_1 + 1\,, \quad n_2' = n_2\,, \quad n_3' = n_3\,,\\
q_{12}' &= \frac{n_1 n_2 \alpha + n_2}{(n_1+1)n_2} = \frac{n_1\alpha + 1}{n_1 + 1}\,,\\
q_{23}' &= \beta\,,\\
q_{31}' &= \frac{n_1\gamma}{n_1+1}\,,
\end{aligned}$$

from which it follows that:

$$q_{12}' + q_{23}' + q_{31}' - 1 = \frac{n_1}{n_1+1}(\alpha+\beta+\gamma-1) + \frac{\beta}{n_1+1}\,. \tag{18}$$

Using (17) we obtain from (18) that:

$$\begin{aligned}
\alpha' + \beta' + \gamma' - 1 &\le \frac{n_1}{n_1+1}[\beta + (1-\beta)\gamma)] + \frac{\beta}{n_1+1}\\
&= \beta + \frac{n_1}{n_1+1}\gamma(1-\beta)\\
&= q_{23}' + q_{31}'(1-q_{23}')\\
&= 1 - (1-q_{23}')(1-q_{31}')\\
&\le 1 - (1-\beta')(1-\gamma')\,.
\end{aligned}$$

Case 2: M_2 is attributed the additional number n. We obtain:

$$\begin{aligned}
n_1' &= n_1\,, \quad n_2' = n_2 + 1\,, \quad n_3' = n_3\,,\\
q_{12}' &= \frac{n_2\alpha}{n_2+1}\,,\\
q_{23}' &= \frac{n_2 n_3 \beta + n_3}{(n_2+1)n_3} = \frac{n_2\beta+1}{n_2+1} = \beta + \frac{1-\beta}{n_2+1} \ge \beta\,,\\
q_{31}' &= \gamma\,,
\end{aligned}$$

from which it follows that:

$$q_{12}' + q_{23}' + q_{31}' - 1 = \frac{n_2}{n_2+1}(\alpha+\beta+\gamma-1) + \frac{\gamma}{n_2+1}\,. \tag{19}$$

Using (17) we obtain from (19) that:

$$\begin{aligned}
\alpha' + \beta' + \gamma' - 1 &\leq \frac{n_2}{n_2+1}[\gamma + (1-\gamma)\beta)] + \frac{\gamma}{n_2+1} \\
&= \gamma + \frac{n_2}{n_2+1}\beta(1-\gamma) \\
&\leq 1 - (1-\beta)(1-\gamma) \\
&\leq 1 - (1-q'_{23})(1-q'_{31}) \\
&\leq 1 - (1-\beta')(1-\gamma')\,.
\end{aligned}$$

<u>Case 3</u>: M_3 is attributed the additional number n. We obtain:

$$\begin{aligned}
n'_1 &= n_1\,, \quad n'_2 = n_2\,, \quad n'_3 = n_3 + 1\,, \\
q'_{12} &= \alpha\,, \\
q'_{23} &= \frac{n_3\beta}{n_3+1}\,, \\
q'_{31} &= \frac{n_3\gamma + 1}{n_3+1} = \gamma + \frac{1-\gamma}{n_3+1} \geq \gamma\,,
\end{aligned}$$

from which it follows that:

$$q'_{12} + q'_{23} + q'_{31} - 1 = \frac{n_3}{n_3+1}(\alpha + \beta + \gamma - 1) + \frac{\alpha}{n_3+1}\,. \tag{20}$$

Using (17) we obtain from (20) that:

$$\begin{aligned}
\alpha' + \beta' + \gamma' - 1 &\leq \frac{n_3}{n_3+1}[1 - (1-\beta)(1-\gamma)] + \frac{\alpha}{n_3+1} \\
&= \frac{n_3}{n_3+1} - \frac{n_3}{n_3+1}(1-\gamma) + \frac{n_3\beta}{n_3+1}(1-\gamma) + \frac{\alpha}{n_3+1} \\
&= \frac{n_3}{n_3+1} + (\frac{1}{n_3+1} - \frac{n_3+1}{n_3+1})(1-\gamma) + q'_{23}(1-\gamma) + \frac{\alpha}{n_3+1} \\
&= 1 - (1-q'_{23})(1-\gamma) + \frac{\alpha-\gamma}{n_3+1} \\
&\leq 1 - (1-q'_{23})(1-q'_{31}) \\
&\leq 1 - (1-\beta')(1-\gamma')\,.
\end{aligned}$$

Note that the new 123-loop with weights q'_{12}, q'_{23}, q'_{31} is not necessarily of type (8).

In the three cases, we have reproduced the required upper bound for $\alpha' + \beta' + \gamma' - 1$, but the proof that the new standard triplet generates a dice-transitive relation is not yet complete. Indeed, the upper bound condition must still be verified for the reverse loop-direction. However, since we can arbitrarily give names to the multisets, we can formally interchange the names of M_2 and M_3 in the above proof in order to obtain the proof for the reverse loop, which is also of type (8).

Finally, we still need to start the induction and therefore have to consider the basic case, which according to the induction hypothesis consists of a standard triplet where the multiset containing the highest number n is a singleton. We need to prove that for such a triplet, inequality (17) holds for both loop-directions. Again three cases must be considered, depending on whether M_1, M_2 or M_3 is the singleton containing n. In all three cases it turns out that for both loop-directions it holds that $\alpha = 0$ and $\gamma = 1$. Therefore, it is sufficient that for both loop-directions $0 + \beta + 1 - 1 \leq 1$, but this inequality is indeed always satisfied. ∎

Dice-transitivity is a weaker type of transitivity than $T_{\mathbf{P}}$-transitivity, but is stronger than $T_{\mathbf{L}}$-transitivity. This follows from the fact that

$$U_{\mathbf{P}}(\alpha, \beta, \gamma) = \beta + \alpha(1 - \beta) \leq \beta + \gamma(1 - \beta) = U_D(\alpha, \beta, \gamma)\,,$$

and

$$U_D(\alpha, \beta, \gamma) = 1 - (1 - \beta)(1 - \gamma) \leq 1 = U_{\mathbf{L}}(\alpha, \beta, \gamma)\,.$$

Similarly, since $U_{ms} \leq U_D$, it also holds that moderate stochastic transitivity implies dice-transitivity.

It is an interesting result that under the same conditions as for $T_{\mathbf{P}}$-transitive probabilistic relations, also dice-transitive probabilistic relations can be generated by a dice model.

Theorem 5.6 *Every three-dimensional dice-transitive probabilistic relation Q with rational elements can be generated by a dice model.*

Proof: The proof closely resembles the proof of Theorem 5.4. We again consider the case where (8) holds, so that

$$q_{12} = \alpha_{123},\ q_{23} = \beta_{123},\ q_{31} = \gamma_{123}.$$

and let $n,\ p,\ q,\ r$ denote the same quantities as before. Hence, with these notations dice-transitivity means that the double inequality

$$pq \leq n(p + q + r - n) \leq n(q + r) - qr$$

holds.

Since $pq \leq nq \leq n(q+r)-qr$, we can again distinguish two cases for the construction of the standard triplet (M_1, M_2, M_3). The first case is the one where $p,\ q,\ r$ satisfy:

$$pq \leq n(p+q+r-n) \leq nq\,,$$

or, equivalently:

$$(n-p)(n-q) \leq nr \leq n(n-p)\,. \tag{21}$$

Then we define:

$$\begin{aligned} M_1 &= E^c \cup \mathbb{N}[3n-p+1, 3n]\,, \\ M_2 &= \mathbb{N}[n-p+q+1, 2n-p+q]\,, \\ M_3 &= E \cup \mathbb{N}[2n-p+q+1, 3n-p]\,, \end{aligned} \tag{22}$$

with E a q-dimensional subset of $\mathbb{N}[1, n-p+q]$ and $E^c = \mathbb{N}[1, n-p+q] \setminus E$. Note that E^c is $(n-p)$-dimensional. From (22) it is immediately clear that $D(M_1, M_2) = np/n^2 = \alpha_{123}$ and $D(M_2, M_3) = nq/n^2 = \beta_{123}$. Depending upon the choice of E we obtain that $D(M_3, M_1)$ can vary in steps of $1/n^2$ from $(n-p)(n-q)/n^2$ when $E = \mathbb{N}[1, q]$ to $n(n-p)/n^2$ when $E = \mathbb{N}[n-p+1, n-p+q]$. Hence for all r satisfying (21) at least one subset E can be found for which $D(M_3, M_1) = nr/n^2 = \gamma_{123}$.

The second case is the one where $p,\ q,\ r$ satisfy:

$$nq \leq n(p+q+r-n) \leq n(q+r)-qr\,,$$

or, equivalently:

$$n(n-r) \leq np \leq n^2 - qr\,. \tag{23}$$

We define:

$$\begin{aligned} M_1 &= E \cup \mathbb{N}[2n+r+1, 3n]\,, \\ M_2 &= E^c \cup \mathbb{N}[2n-q+r+1, 2n+r]\,, \\ M_3 &= \mathbb{N}[n-q+r+1, 2n-q+r]\,, \end{aligned} \tag{24}$$

with E an r-dimensional subset of $\mathbb{N}[1, n-q+r]$ and $E^c = \mathbb{N}[1, n-q+r] \setminus E$. E^c is $(n-q)$-dimensional. From (24) it immediately follows that $D(M_2, M_3) = nq/n^2 = \beta_{123}$ and $D(M_3, M_1) = nr/n^2 = \gamma_{123}$. Depending upon the choice of E we obtain that $D(M_1, M_2)$ can vary from $n(n-r)/n^2$ when $E = \mathbb{N}[1, r]$ to $(n(n-r)+r(n-q))/n^2 = (n^2-qr)/n^2$ when $E = \mathbb{N}[n-q+1, n-q+r]$. In particular, for all p satisfying (23) at least one subset E can be found for which $D(M_1, M_2) = np/n^2 = \alpha_{123}$. ∎

Note that since $T_{\mathbf{P}}$-transitivity implies dice-transitivity, a three-dimensional $T_{\mathbf{P}}$-transitive probabilistic relation with rational elements can also be generated by the standard triplets constructed in Theorem 5.6 which are in general different from the standard triplets used in Theorem 5.4.

6 Toward higher dimensions

In [15], besides the utility model, which yields probabilistic relations with strong transitivity properties, also the so-called multidimensional model is discussed. Moreover, it has been shown that the probabilistic relations generated by this multidimensional model are $T_{\mathbf{L}}$-transitive, and conversely, all $T_{\mathbf{L}}$-transitive probabilistic relations on a universe of dimension $n \leq 5$ can be generated by a multidimensional model. By analogy, as far as our model is concerned, the question arises whether the reverse property, which has been proven in Theorem 5.6 to hold for three-dimensional probabilistic relations with rational elements, extends to higher-dimensional probabilistic relations. The question must be answered in negative sense, as follows from:

Theorem 6.1 *Not all four-dimensional dice-transitive probabilistic relations can be generated by a dice model.*

Proof: We will construct a set of graphs that exhibit the dice-transitivity of the associated probabilistic relation but for which there does not exist a standard quartet (M_1, M_2, M_3, M_4) that generates it. We will use the graph of Figure 2, which shows explicitly that $D(M_1, M_3) = e = 0$ and $D(M_2, M_4) = f = 0$. Obviously, it holds that a, b, c, $d \in [0, 1]$.

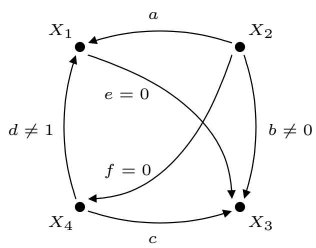

Figure 2: Dice-transitive probabilistic relations that cannot be generated by a dice model.

In this graph there are four subgraphs with 3 nodes. The dice-transitivity has to hold for each subgraph. We therefore have the following four conditions that must hold:

$$\begin{cases} 0 \le d-a \le 1-a(1-d) & \text{, for triplet } (M_1, M_2, M_4) \\ 0 \le d-c \le 1-c(1-d) & \text{, for triplet } (M_1, M_3, M_4) \\ 0 \le c-b \le 1-b(1-c) & \text{, for triplet } (M_2, M_4, M_3) \\ 0 \le a-b \le 1-b(1-a) & \text{, for triplet } (M_2, M_1, M_3) \end{cases}$$

which is equivalent to

$$\begin{cases} b \le c \le d\,, \\ b \le a \le d\,. \end{cases} \tag{25}$$

Note that these conditions can easily be satisfied. We now prove that, when $e = 0$, $f = 0$, when the conditions (25) are fulfilled and when

$$b \neq 0,\ \ d \neq 1\,, \tag{26}$$

we have examples of graphs that are dice-transitive but which cannot be generated by a standard quartet.

Let us assume that there does exist a standard quartet (M_1, M_2, M_3, M_4) with these properties and let $a_1 = \max M_1$ and $a_2 = \max M_2$. We have two cases. In the first case we have $a_1 > a_2$ from which it follows that $b = 0$. In the second case we have $a_1 < a_2$ from which it follows that $d = 1$. In the first case we used the fact that $e = 0$ and in the second case that $f = 0$. These two cases represent all possible situations and (26) does not hold in either case. Therefore, there exist no standard quartets that correspond to the dice-transitive graph having the following properties:

$$\begin{cases} b \le c \le d\,, \\ b \le a \le d\,, \\ b \neq 0\,, \\ d \neq 1\,, \\ e = 0\,, \\ f = 0\,. \end{cases} \tag{27}$$

Again, the conditions (27) can easily be satisfied. ∎

7 Conclusions

We have developed a new model for generating probabilistic relations of which the transitivity properties can vary from strong $T_{\mathbf{M}}$-transitivity to a weak form of cycle-transitivity situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity and which we called dice-transitivity. A complete characterization of the m-dimensional probabilistic relations generated by the dice model has till now not been found for $m > 3$.

This new type of transitivity, in which relations that show a cyclic behaviour are not necessarily excluded, is confirmed as a characteristic property when comparing distributions of independent random variables, using generalized dice models [7]. Recently, we have shown that when comparing distributions belonging to a same parametric family, such as exponential distributions, geometric distributions, uniform distributions on intervals of fixed length, Laplace distributions with the same variance or normal distributions with the same variance, more specific types of cycle-transitivity are encountered [7]. It is envisaged that the exploitation of this inherent transitivity may lead to augmented statistical procedures related to the comparison of random variables.

Finally, the proposed model is a rich source for many interesting questions on a variety of combinatorial properties of collections of multisets. We will report on these properties in the near future.

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