Comparison of Random Variables from a Game-Theoretic Perspective

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Preface

When I arrived at this department, on December 1st 2001, my mind was troubled with uncertainty and doubt, although lacking fear. From the first day, however, I was warmly welcomed by my colleagues and that day thus ended with relief. Arriving here as a teaching assistant, I was yet without research subject. Hans De Meyer was suggested to me by a few colleagues, and so I went to his office for a small talk. Not knowing much about me, he was still eager to support me as supervisor, even more, he let me take part in his own exciting research, done together with Bernard De Baets. Bernard, knowing me even less, also generously accepted to be my supervisor. I now had two supervisors and a research playing ground called "The Dice Model," and the results of this play can be found in this work. I am very grateful for their given support and interest, and my scientific mind has improved significantly due to the interaction with my supervisors.

Of all colleagues, Glad Deschrijver, with whom I've shared the office for over three years, was most friendly and helpful. I am very lucky to have had such a fine colleague, with whom I could share the joy of mathematical exploration. The remembrance of the many discussions on varying topics will always remain. His LaTeX expertise, which he shared unconditional, has been very helpful throughout the years and is strongly appreciated. Of international researchers, I especially enjoyed the company of János Fodor, Michel Grabisch and Sándor Jenei, furthermore, it was a suggestion of Sándor that led to the investigation of the game variant discussed in Chapter 5.

None of this would have been possible, of course, without the support and love — and patience — of my family. Although it is a cliché, I owe my mother and father much gratitude. The realism of my brother and his friend Stephan has always been helpful and appreciated and it has kept me in touch with the real world. My friends that have lasted through the years, and on which I could always rely, should be praised for putting up with me. My girlfriend Annemiek should also be admired for that reason, and I am very happy she lets me be a part of her life.

Finally, I wish to thank the Open Source Community, of which I feel a part, for making great pieces of software freely and openly available, and for handing me a fruitful, interesting, useful and satisfying hobby.

Bart De Schuymer,
May 30, 2005.

Overview

In this work, the comparison of random variables by means of a generated probabilistic relation is the central theme. The comparison scheme that will be used throughout this work will be introduced in Chapter 3 and essential to this scheme is the probability that one random variable takes a greater value than the other random variable. Put in game-theoretic context (when the random variables are associated to players), this probability can also be defined as the probability that one player wins from the other. The random variables are therefore compared from a game-theoretic perspective. Even more, it is very natural to transform the elements of the probabilistic relation obtained by using this comparison scheme into the elements of a payoff matrix, to which can be associated the games that will be discussed in Chapters 5 and 7.

In Chapter 1, basic concepts are introduced that will be needed in the subsequent chapters. In the first section, the relational concepts and related subjects that will be encountered in this work are introduced. The second section introduces distribution functions, states their connection to copulas and provides an important note on random vectors. The third section then introduces the game-theoretical notions that will be needed in Chapters 5 and 7. In the last section of the first chapter, some basic notions from partition theory that will be of use in Chapters 5, 7 and 8, are introduced.

Chapter 2 introduces the framework of cycle-transitivity, which is ideally suited for describing and comparing forms of transitivity. As many probabilistic relations will be encountered throughout this work, including relations that are not necessarily transitive in the strict sense (such as dice-transitive relations), a means to represent these relations in a uniform way is needed. The framework of cycle-transitivity is ideal for this because the transitivitiy of diverse types of probabilistic relations can be represented in this model, including non-transitive relations. Cycle-transitivity is a way of describing a 3-dimensional probabilistic relation by means of the cyclic evaluation of the weights of the corresponding weighted graph and it will be omnipresent in this work. In the first two sections, this framework is introduced. Sections three and four then consider the representation in cycle-transitivity notation of the two best-known types of transitivity, namely fuzzy transitivity and stochastic transitivity. In the course of the discussion, new types of transitivity, such as partial g-stochastic transitivity and isostochastic transitivity, will be defined. The last section then briefly discusses an alternative representation of probabilistic relations by using so-called symmetric payoff relations.

In Chapter 3, the discrete dice model is introduced and its characteristic transitivity, called dice-transitivity, is determined. This model can be used to pairwisely compare lists of integers and these lists can be regarded as fair dice with each integer from the list written on a face of the dice. This model will be used in Chapter 5 as the model in which an interesting class of games, called (n, σ) dice games, is defined. It is also the basis of the general comparison scheme for random variables that will be investigated in Chapters 4 and 6. The first two sections introduce the dice model while the third section unfolds the importance of a so-called standard collection of lists. In the fourth section dice-transitivity is proven to be the characteristic transitivity generated by a dice model. Section 5 then answers a natural question concerning the nature of dice-transitivity and its connection to the probabilistic sum. Furthermore, it is proven that any 3-dimensional dice-transitive relation with rational elements can be generated by a 3-dimensional dice model consisting of at most seven so-called blocks, which implies that the dice model used to generate a given 3-dimensional dice-transitive relation can be chosen to have a much simpler structure than that of a general dice model. The transitivity of higher-dimensional dice models is investigated in the sixth section. It is proven that not all 4-dimensional dice-transitive relations with rational elements can be generated by 4-dimensional dice models, implying that dice-transitivity does not remain the characteristic transitivity of higher-dimensional dice models. Despite the undertaking of various attempts to find out the characteristic transitivity of 4-dimensional dice models, we were unable to pin-point it. It will be shown, however, that all 4-dimensional $T_{\mathbf{M}}$-transitive probabilistic relations with rational elements can be generated by a 4-dimensional dice model.

In Chapter 4, the method introduced in the previous chapter is generalized to obtain a mathematical tool to pairwisely compare random variables. This new method of comparing independent random variables provides a graded alternative to the concept of stochastic dominance, which is very popular in the field of decision making, e.g. in economical applications. As this work is primarily mathematical, we will not go into further detail about these applications and merely refer to the cited references. The new comparison method that is introduced in this chapter provides a promising alternative to the often hard conditions for stochastic dominance. The first section generalizes the dice model. Depending on whether discrete or continuous random variables are compared, these models are called generalized discrete dice models or generalized continuous dice models. These models all generate probabilistic relations and when comparing independent random variables these relations are all at least dice-transitive. The need to pairwisely compare random variables occurs frequently in the field of decision making. In Chapter 4, the emphasis is laid on the comparison of random variables by considering them to be independent. Using a copula differing from the $T_{\mathbf{P}}$-copula to pairwisely bind the random variables is postponed to Chapter 6. In the second section of Chapter 4 it is proven that the characteristic transitivity of the generalized dice models with independent random variables remains dice-transitivity. The remainder of the chapter then considers more specific families of random variables. An emphasis is laid on the types of cycle-transitivity of the relations that can be generated by these models. New types of transitivity and interesting connections to the field of copulas are laid bare. The third section discusses the general class of dice models consisiting of independent random variables for which the cumulative distribution functions are arbitrary translations of the same c.d.f. Section 4 then considers various one-parameter families of random variables and multiple types of transitivity, including multiplicative transitivity and specific types of isostochastic transitivity, will be encountered in that section. Section 5 considers dice models consisting of normally distributed independent random variables where both the expected value and variance are left as parameters. It is shown that moderate stochastic transitivity is the characteristic transitivity of such 3-dimensional models. In the final section dice models consisting of uniformly distributed independent random variables with overlapping support are considered and the characteristic transitivity of such 3-dimensional models is determined, which will turn out to be a type of g-stochastic transitivity.

Chapter 5 is devoted to obtaining the optimal strategies of games that are closely connected to the dice model that was introduced in Chapter 3. The games are symmetric matrix games that are played between two persons who have a collection of dice with a fixed number of faces and with strictly positive integers written on the faces summing up to a fixed number to their disposition. The games considered in Chapter 7 will be closely related to those considered in Chapter 5. In their statistical interpretation, the difference between the considered games in both chapters will be the copula that is used to define the payoff function. Chapter 5 considers the game variant in which the $T_{\mathbf{P}}$-copula is used. Finding the optimal strategies for this game variant is therefore closely related to the comparison of independent uniformly distributed random variables. The first three sections of Chapter 5 are concerned with giving a full description of the considered game variant. Section 4 then bundles the answers to the following questions concerning the nature of the optimal strategies: which (n, σ) dice games contain optimal strategies, how do the optimal strategies look and how many are they ? Section 5 then proves the results stated in the preceding section.

In Chapter 6, the method to pairwisely compare random variables, introduced in Chapter 4, is considered when the random variables are compared using a copula different from the $T_{\mathbf{P}}$-copula. An emphasis is laid on the two extreme copulas. The first section introduces an alternative method to obtain the probabilistic relation generated by a discrete dice model using the so-called diagonal formula. Moreover, an equivalent representation using ordered lists is obtained for dice models consisting of discrete random variables pairwisely coupled by one of the two extreme copulas. This representation forms the link to the games considered in Chapter 7 and this ordered list representation is also used in the second section to determine the characteristic transitivity for these models. It is proven that $T_{\mathbf{L}}$-transitivity (resp. partial min-stochastic transitivity) is the characteristic transitivity of 3-dimensional dice models consisting of discrete random variables that are, for comparison reasons, pairwisely coupled by the $T_{\mathbf{M}}$-copula (resp. $T_{\mathbf{L}}$-copula). It is also proven that for neither type of the considered dice models, the specific transitivity is maintained when considering higher-dimensional models. The third section then considers continuous dice models in which one of the extreme copulas is used for comparing the random variables. For both types of models, an interesting way of determining the probabilistic relation is obtained using the graphs of the marginal cumulative distribution functions corresponding to the considered random variables.

Chapter 7 considers two game variants that have, apart from the used copula, the same definition as the game variant considered in Chapter 5. The first section gives a brief overview of the three game variants that are encountered in this work. The two subsequent sections then discuss the game variant in which the $T_{\mathbf{M}}$-copula is used. In the second section, the results about the optimal strategies are bundled and they are then proven in the third section. Finally, sections 4 and 5 discuss the game variant in which the $T_{\mathbf{L}}$-copula is used. The first of these sections again bundles the results, while in the last section these results are proven. It turns out that, although the definitions of the game variants only differ by the used copula, the characterization of the optimal strategies is completely different for each variant.

In Chapter 8, standard n-duplets and n-triplets, which are certain collections of mutually disjoint sets of strictly positive integers, are partitioned using their so-called street number with the aim of determining how many such collections have a given street number. Section 1 introduces the street number and shows its connection to the payoff matrix of the games defined in Chapter 5. The second section then determines, for a given street number, how many n-duplets have this street number. It turns out that certain concepts from partition theory are needed to solve this problem. The third section then introduces the so-called dual partition set, which leads to an interesting method to construct "rectangular triangles" by means of rectangles. Section 4 then briefly considers the street number of n-triplets. It is indicated that it becomes very difficult to give general results concerning the number of n-triplets having a given street number.

1 Introduction

1.1 Relational concepts

1.1.1 Probabilistic and fuzzy relations

In this work probabilistic relations, often also called reciprocal or ipsodual relations, play a central role. Probabilistic relations serve as a popular representation of various relational preference models [15, 38, 73].

DEFINITION – 1.1.1: *A probabilistic relation Q on a set of alternatives A is an $A^2 \to [0,1]$ mapping such that for all a, b it holds that*

$$Q(a,b) + Q(b,a) = 1 \,.$$

If A is finite with cardinality m, then Q is called an m-dimensional probabilistic relation.

The number $Q(a,b)$ can, for instance, express the degree of preference of alternative a over alternative b. When a crisp model is used, a subject is confronted with 2 alternatives A and B and is asked which one is preferred. There are 3 possible answers: "A is preferred to B," "B is preferred A" or "A and B are equally preferred (indifference)." Throughout this work, we will never consider probabilistic relations in which incomparability can occur. In a probabilistic model a subject or subjects is or are asked multiple times which one is preferred. The proportion of answers in which A is preferred with respect to all – say n – answers may be treated as the degree to which A is preferred to B:

$$P(A,B) = \frac{\#\{A \text{ is preferred to } B\}}{n} \,.$$

As indifference is allowed, the relation $P(A,B)$ is not necessarily a probabilistic relation. However, a probabilistic relation can easily and intuitively be obtained:

$$Q(A,B) = P(A,B) + \frac{1}{2} I(A,B) \,,$$

where

$$I(A,B) = \frac{\#\{A \text{ and } B \text{ are equally preferred}\}}{n} \,.$$

It is easily verified that $Q(A,B)$ is a probabilistic relation. Note that a subject need not necessarily be asked her preference multiple times. For example, the statement "I prefer apples to bananas" can be taken to imply that, given the choice, I will select apples more often than I select bananas, but occasionally I may select bananas. An estimate of the size of my preference could come from observing such choices and by summarizing them as a ratio, a proportion, or a probability [72].

We can also use a fuzzy model of preferences to obtain a probabilistic relation by using a fuzzy preference relation R. For any 2 alternatives A and B it must then hold that $R(A,B) + R(B,A) = 1$, and in this model, the subject is asked to what extent alternative A is preferred to alternative B. The answer is then a number, say $R(A,B)$, located in the unit interval $[0,1]$ and it expresses the subjective judgment of degree in which A is preferred to B. When $R(A,B)=0$, B is completely preferred to A, while $R(A,B)=1$ indicates that A is completely preferred to B. On the other hand, when $R(A,B)=1/2$, A and B are equally preferred. When $R(A,B)\in\,]1/2,1[$, the value indicates a partial (fuzzy) degree of preference of A over B. Note that the property $R(A,B)+R(B,A)=1$ is in general not always required for a fuzzy preference relation, but in this work only fuzzy preference relations for which this quite natural property holds will be encountered. Finally, note that fuzzy preference relations are a subclass of the more general type of relations called fuzzy relations. A fuzzy relation R on A is an $A^2 \to [0,1]$ mapping that expresses the degree of relationship between elements of A: $R(a,b)=0$ means a and b are not related at all, $R(a,b)=1$ expresses full relationship, while $R(a,b)\in\,]0,1[$ indicates a partial degree of relationship only.

Of course, specific probabilistic relations can also be defined by theoretical models instead of being directly obtained from questioning a subject. In this work we will concentrate on such theoretical models. For such models, it is often interesting to study which kind of transitive relations can be generated by them. This gives an idea of how restrictive the model is, which probabilistic relations can be modelled by them, and it can also show connections with other models. Transitivity properties are described using specific classes of aggregation operators, which we will now introduce.

1.1.2 Aggregation operators

The number of well-known types of operators that will be used in this work is quite limited. However, transitivity will play a crucial role and it is therefore not surprising that we will need the concept of the so-called triangular norm (briefly t-norm), which is used for describing the transitivity of fuzzy relations.

DEFINITION – 1.1.2: [70] *A binary operation $T : [0,1]^2 \to [0,1]$ is called a t-norm if it satisfies:*

(i) Neutral element 1*:* $(\forall x \in [0,1])(T(x,1)=T(1,x)=x)$.

(ii) Monotonicity: T is increasing in each variable.

(iii) Commutativity: $(\forall (x,y)\in[0,1]^2)(T(x,y)=T(y,x))$.

(iv) Associativity: $(\forall (x,y,z)\in[0,1]^3)$
$(T(x,T(y,z))=T(T(x,y),z))$.

A related concept is that of a t-conorm, which is a binary operation on $[0,1]$ satisfying the above conditions (ii)–(iv) and which has as neutral element 0. Also, to any t-norm T corresponds a dual t-conorm S defined by

$$S(x,y)=1-T(1-x,1-y)\,. \tag{1.1}$$

For a recent monograph on t-norms and t-conorms, we refer to [56].

The smallest t-norm is the drastic product $T_{\mathbf{D}}$, which is right-continuous only and is 0 everywhere up to the boundary condition $T_{\mathbf{D}}(x,1) = T_{\mathbf{D}}(1,x) = x$. The three main continuous t-norms are the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$. These three t-norms belong to one of the most important parametric t-norm families, namely the Frank family $(T^{\mathbf{F}}_{\lambda})_{\lambda\in[0,\infty]}$ [41], which turns out to be also a family of copulas (see below). For $\lambda \in]0,1[\cup]1,\infty[$, the t-norm $T^{\mathbf{F}}_{\lambda}$ is defined by

$$T^{\mathbf{F}}_{\lambda}(x,y) = \log_{\lambda}\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) . \tag{1.2}$$

As limit cases, one obtains $T_{\mathbf{M}}$ ($\lambda \to 0$), $T_{\mathbf{P}}$ ($\lambda \to 1$) and $T_{\mathbf{L}}$ ($\lambda \to \infty$).

On the other hand, in this work a crucial role is played by random variables and their interdependence. In this respect the concept of a copula will show up in Chapter 6. Also, throughout Chapters 2 and 4, we will encounter numerous examples of copulas in the context of describing the transitivity of specific probabilistic relations.

DEFINITION – 1.1.3: [2, 46, 64] *A binary operation* $C : [0,1]^2 \to [0,1]$ *is called a quasi-copula if it satisfies:*

(i) Neutral element 1*:* $(\forall x \in [0,1])(C(x,1) = C(1,x) = x)$.

(i') Absorbing element 0*:* $(\forall x \in [0,1])(C(x,0) = C(0,x) = 0)$.

(ii) Monotonicity: C is increasing in each variable.

(iii) 1-Lipschitz property: $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$(|C(x_1,y_1) - C(x_2,y_2)| \le |x_1 - x_2| + |y_1 - y_2|) .$$

If instead of (iii), C satisfies

(iv) Moderate growth (2-increasing): $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$((x_1 \le x_2 \,\wedge\, y_1 \le y_2) \Rightarrow C(x_1,y_1) + C(x_2,y_2) \ge C(x_1,y_2) + C(x_2,y_1)) ,$$

then it is called a copula.

Note that condition (i') follows from conditions (i) and (ii). For a copula, condition (ii) can be omitted as it follows from (iv) and (i'). As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is not true.

It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. Finally, note that for any quasi-copula C it holds that $T_{\mathbf{L}} \le C \le T_{\mathbf{M}}$.

The importance of copulas in this work is partially due to their relation with bivariate cumulative distribution functions, which will be discussed in the next section.

1.1.3 Transitivity of fuzzy and probabilistic relations

When a person says she prefers apples to bananas and bananas to cherries, this can be taken to imply that she will prefer apples to cherries, meaning the preference relation is transitive. The size of the preference shown for apples over cherries in relation to the other two preferences defines the degree of transitivity found, and this determines how precisely one can predict untested choices [72].

Whatever relational representation is employed for intensities of preference, transitivity is always an interesting, often desirable property. In the context of fuzzy preference modelling, for instance, T-transitivity of fuzzy (i.e. $[0,1]$-valued) relations is an indispensable notion [9, 14, 40, 68]. Some types of transitivity have been devised specifically for probabilistic relations, such as various types of stochastic transitivity [38, 63, 67].

Transitivity is a simple, yet powerful property of relations. A (binary) relation R on a universe A (often referred to as the set of alternatives) is called transitive if for any $(a,b,c) \in A^3$ it holds that

$$((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R\,. \tag{1.3}$$

Identifying a relation with its characteristic mapping, i.e. defining

$$R(a,b) = \begin{cases} 1 & \text{, if } (a,b) \in R\,, \\ 0 & \text{, if } (a,b) \notin R\,, \end{cases}$$

transitivity can be stated equivalently as

$$(R(a,b) = 1 \wedge R(b,c) = 1) \Rightarrow R(a,c) = 1\,.$$

However, many other equivalent formulations may be devised, such as

$$(R(a,b) \geq \alpha \wedge R(b,c) \geq \alpha) \Rightarrow R(a,c) \geq \alpha\,, \tag{1.4}$$

for any $\alpha > 0$. Alternatively, transitivity can also be expressed in the following functional form:

$$\min(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{1.5}$$

Note that on $\{0,1\}^2$ the minimum operation is nothing else but the Boolean conjunction.

Transitivity of relations is closely connected to consistency. Suppose, e.g., that the relation represents the property "is preferred to," which is the relation that will be used throughout this work. It is then natural to demand that if a is preferred over b and b is preferred over c, then a should be preferred over c. In other words, it is natural to demand that the preference relation is transitive. However, in this work we will encounter models for probabilistic relations Q in which weak stochastic transitivity, given by (1.7), is not even satisfied. In experimental tests also, it has been observed that when persons are asked to pairwise express their preference between the elements of some set, it is not uncommon that the resulting preference relation is not transitive. In other words, cycles may occur (as incomparability is not allowed in this work). In this respect, the framework of cycle-transitivity, to be introduced in the next chapter, is very convenient as, unlike most other frameworks, it does not exclude the possibility of cyclic behavior. When confronted with a specific probabilistic model, we are often interested in the types of transitivity the relations generated by the model can possess. Probabilistic relations can then be classified on the basis of their type of transitivity. For various relations, types of transitivity have been defined that are a natural expansion of the many equivalent definitions of transitivity on $\{0,1\}$-valued relations.

1.1.3.1 Transitivity of fuzzy relations

In the setting of fuzzy set theory, in which relations need not be reciprocal, formulation (1.5) has led to the popular notion of T-transitivity, where a t-norm T is used as a generalization of the Boolean conjunction.

DEFINITION – 1.1.4: *Let T be a t-norm. A fuzzy relation R on A is called T-transitive if for any $(a,b,c) \in A^3$ it holds that*

$$T(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{1.6}$$

1.1.3.2 Transitivity of probabilistic relations

Transitivity properties for probabilistic relations rather have the logical flavor of expression (1.4). There exist various kinds of stochastic transitivity for probabilistic relations [15, 63]. For instance, a probabilistic relation Q on A is called weakly stochastic transitive if for any $(a,b,c) \in A^3$ the following implication, which corresponds to the choice of $\alpha = 1/2$ in (1.4), is satisfied:

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq 1/2\,. \tag{1.7}$$

Next, let R be a complete ($\{0,1\}$-valued) relation on A, which means that $\max(R(a,b), R(b,a)) = 1$ for any $a,b \in A$. It then holds that R has an equivalent $\{0,1/2,1\}$-valued probabilistic representation Q given by

$$Q(a,b) = \begin{cases} 1 & \text{, if } R(a,b) = 1 \text{ and } R(b,a) = 0\,, \\ 1/2 & \text{, if } R(a,b) = R(b,a) = 1\,, \\ 0 & \text{, if } R(a,b) = 0 \text{ and } R(b,a) = 1\,. \end{cases}$$

Or in a more compact arithmetic form:

$$Q(a,b) = \frac{1 + R(a,b) - R(b,a)}{2}\,. \tag{1.8}$$

One easily verifies that R is transitive if and only if its probabilistic representation Q satisfies, for any $(a,b,c) \in A^3$:

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) = \max(Q(a,b), Q(b,c))\,. \tag{1.9}$$

Similarly, a *weakly complete* fuzzy relation R, i.e. one satisfying

$$R(a,b) + R(b,a) \geq 1\,,$$

for any $a, b \in A$, can be transformed into a (non-equivalent, yet interesting) probabilistic representation $Q = P + I/2$, with P and I the (fuzzy) strict preference and indifference components of the fuzzy preference structure (P, I, J) generated from R by means of $T_{\mathbf{L}}$ [21, 79]:

$$\begin{aligned} P(a,b) &= T_{\mathbf{M}}(R(a,b), 1 - R(b,a)) = 1 - R(b,a)\,, \\ I(a,b) &= T_{\mathbf{L}}(R(a,b), R(b,a)) = R(a,b) + R(b,a) - 1\,, \\ J(a,b) &= T_{\mathbf{L}}(1 - R(a,b), 1 - R(b,a)) = 0\,. \end{aligned}$$

Note that the corresponding expression for Q is formally the same as (1.8). For an introduction to fuzzy preference structures, we refer to [20].

Note that T-transitivity for probabilistic relations is conceptually a relatively strong condition compared to the stochastic transitivity variants, which resemble (1.4) (see Subsection 2.4.1). For any triplet $(Q(a,b), Q(b,c), Q(c,a))$ there are possibly 6 conditions to be satisfied, augmented with the condition of Q being a probabilistic relation. For the stochastic transitivity variants, the number of conditions to be satisfied is in general smaller, as those conditions have as prerequisite a condition similar to $\min(Q(a,b), Q(b,c)) \geq 1/2$ (e.g. condition (1.7)).

We end by fixing a notation which will be used throughout this work. When the set of alternatives A is countable, it is possible to label each alternative $a \in A$ with a different index $i \in \mathbb{N}$. For two alternatives $a, b \in A$ with respective labels $i, j \in \mathbb{N}$, we define $q_{ij} = Q(a,b)$. The probabilistic relation $Q = \{Q(a,b) \mid a, b \in A\}$ can then be written as $Q = [q_{ij}]$.

1.1.3.3 Separate evaluation versus joint evaluation

One way of making a preference relation between alternatives is by using a utility function $u(a)$ on the set of alternatives A. To each alternative, a positive value denoting its "utility" is assigned, the higher the value, the higher the utility. Using this scheme, which is a separate evaluation, the importance of an alternative is determined without explicitly comparing it to any other alternative, which can be called a joint evaluation. These utilities can always be rescaled to $[0,1]$. From these utilities, a preference relation between the alternatives can be devised. Depending upon the nature of the given utility values, one defines a different corresponding preference relation [77]. When the utilities are given as a difference scale normalized on $[0,1]$, the following relation is defined:

$$q_{ij} = \frac{1}{2}(1 + u(a_i) - u(a_j))\,. \tag{1.10}$$

In this case, $q_{ij} - 1/2$ can be seen as an intensity of preference of a_i over a_j. On the other hand, when the utilities represent a positive ratio scale, the preference relation is defined as

$$q_{ij} = \frac{u(a_i)}{u(a_i) + u(a_j)} . \tag{1.11}$$

In this case, when $q_{ji} \neq 0$, q_{ij}/q_{ji} indicates a ratio of the preference intensity for a_i to that for a_j. The type of transitivity of the preference relations defined by (1.10) is called additive transitivity and is defined by the condition

$$q_{ij} + q_{jk} + q_{ki} = \frac{3}{2} , \tag{1.12}$$

those defined using (1.11) are characterized by multiplicative transitivity given by

$$\frac{q_{ij}}{q_{ji}} \frac{q_{jk}}{q_{kj}} \frac{q_{ki}}{q_{ik}} = 1 . \tag{1.13}$$

On the other hand, for any probabilistic relations satisfying (1.12) (resp. (1.13)), utility values $u(a_i)$ can be found so that (1.10) (resp. (1.11)) holds.

The question arises whether it is useful to question a subject about her preference over a number of alternatives by letting her pairwisely compare the alternatives. If there are n alternatives, $n(n-1)/2$ comparisons will need to be made. Perhaps it is better to let her assign n utility values and transform them into a preference relation as was done above? It turns out that these two ways of obtaining a preference relation do not necessarily produce the same preference relation. In the literature, studies are found in which the joint versus separate preference shifts are demonstrated [6, 50, 51]. In one such experiment, MBA students were presented two jobs and were asked which job they would accept [6]. The Money Job offered a salary of $85,000 but a salary of $95,000 for other MBAs, the Fair Job offered a salary of $75,000 for both the questioned student and the other MBAs. In the separate evaluation, participants were presented with offers separately and asked on a case-by-case basis whether they would accept each offer. In the joint evaluation, participants were presented with a pair of hypothetical job offers and asked which offer(s), if any, they would accept. In joint evaluation, participants tended to select the Money Job while in separate evaluation the Fair Job was preferred. These so-called preference reversals show that the disparity in compensation and the resulting social comparison effects were more powerful when jobs were evaluated separately and the actual levels of compensation were more powerful when participants could compare job offers. A few theories exist to explain these social comparison effects [8]. A first explanation suggests that these reversals can be explained by norm theory [66]. When individuals are presented with a single item to evaluate, they struggle to make sense of it. To do so, they evoke a set of available, internal referents for comparison and evaluate the item in the context of these referents. When individuals are presented with more than one item to evaluate, the alternatives themselves provide the comparison set for evaluation. It is argued that, in joint evaluation, if there are differences in category membership across the two alternatives, reconciling these category differences will dominate assessment. A second explanation is the so-called want/should proposition [7]. It suggests a tension between what an individual wants to do versus what the individual thinks she should do. The argument is that we often have an emotional desire to engage in behaviors that are inconsistent with the behaviors in which we believe we should engage. Under separate evaluation, lacking a counterbalancing alternative, we lean toward what we want to do. In joint evaluation, on the contrary, we tend to select the most justifiable option – the one that we think we should choose. A third explanation is given by the evaluability hypothesis [49]. It suggests that these preference reversals are driven by differences in the evaluability of attributes and argues that when two options involve trading off between a hard-to-evaluate attribute and an easy-to-evaluate attribute, the former one will have less impact in separate evaluation than in joint evaluation. None of these explanations fully satisfy all occurrences of preference reversal found in experiments, but it is very likely that each one applies to at least some of the encountered occurrences. Therefore, when one wants a subject to compare alternatives, to obtain better results it is better to use a joint evaluation scheme. The best results would be obtained, of course, when the subject has to compare all alternatives at once, but this is sadly enough beyond the skill of any presently known mortal being when the number of alternatives becomes large.

1.1.3.4 Cycles in probabilistic relations

In a probabilistic relation $Q = [q_{ij}]$, it holds that $q_{ij} = q_{ji}$, conceptually denoting that alternatives i and j are equally preferred, if and only if $q_{ij} = 1/2$. It is therefore only natural to define that alternative i is preferred to alternative j if and only if $q_{ij} > 1/2$. The weakest condition a probabilistic relation has to satisfy for being called transitive, is therefore the condition of weak stochastic transitivity given by (1.7). Conceptually, this condition says that if k is not preferred to j and j is not preferred to i, then k should not be preferred to i. Note that this condition implies that if i and j are equally preferred ($q_{ij} = 1/2$) and j and k are equally preferred ($q_{jk} = 1/2$), then i and k must be equally preferred too ($q_{ik} = 1/2$), which is a condition that is very natural to demand. Also note that weak stochastic transitivity excludes the possibility of i not being preferred over k ($q_{ik} \leq 1/2$), while i is preferred to j but k is not preferred to j ($q_{ij} > 1/2 \wedge q_{kj} \leq 1/2$); again a very natural condition.

When determining the types of transitivity probabilistic relations generated by a specific model can possess, we are really characterizing the probabilistic relations between 3 alternatives i, j and k, given by (q_{ij}, q_{jk}, q_{ki}). We will encounter relations generated by models that are not necessarily even weakly stochastic transitive, meaning that they can contain cycles and are therefore not transitive according to the definition introduced above. However, if we can characterize the probabilistic relations between the 3 alternatives, we will still call this characterization the type of transitivity. For example, the condition for dice-transitivity, given by

$$1 - \min(q_{ij}, q_{jk}, q_{ki}) \geq \operatorname{median}(q_{ij}, q_{jk}, q_{ki}) \max(q_{ij}, q_{jk}, q_{ki}) \, , \forall(i, j, k) \, ,$$

is a weaker condition than that of weak stochastic transitivity, but we will still refer to the above condition as dice-transitivity (although the condition does not impose transitivity as was defined above). Therefore, relations can be dice-transitive, without being transitive.

When preference relations are obtained by questioning persons or studying animal behavior, cycles are not really expected. However, recent studies show that they occur relatively frequently. In [74], 44 students were asked to give their preferences over 5 holiday destinations. It was found that only 79.6 percent of the students produced preference relations without cycles. A similar experiment, with similar results, was discussed in [43]. In that paper, it is shown that the number of alternatives has a great impact on the transitivity of the relations. When increasing the number of alternatives from 4 up to 7, the percentage of students that produced preference relations without cycles decreased from 92 to 50 percent. Possible explanations for these quite remarkable results are not presented in the cited papers. Note that the percentage of cycles compared to the total number of triplets is much lower. The maximum number of cycles of length 3 that are possible when comparing n elements without indifference or incomparability, is given by $(n^3 - n)/24$ if n is odd and $(n^3 - 4n)/24$ if n is even [55]. When there is no indifference, the number c of cycles of length 3 can be computed as

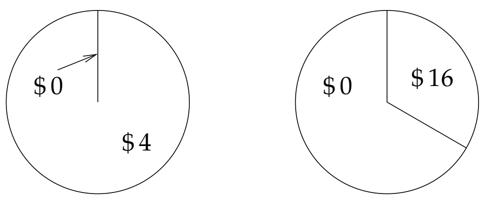
$$c = \frac{1}{6}n(n-1)(n-2) - \frac{1}{2}\sum_{i=1}^{n} s_i(s_i - 1)\,,$$

where n is the number of alternatives and s_i is the outdegree of node i, when the preference relation is represented as a digraph with n nodes and $n(n-1)/2$ directed arcs [45].

Whether the cycles observed in the above experimental results have a logical explanation or not, the occurrence of cycles is certainly a natural phenomenon. They occur, for example, regularly in game theory. Probably the simplest example being the classical rock-paper-scissors game: rock wins from scissors, scissors wins from paper and paper wins from rock. The corresponding preference relation is therefore cyclic (a is preferred to b if a wins from b).

We end by noting that, when obtaining preference relations from individuals, one must be careful not to generalize too much. For example, one must be cautious for preference reversals, which occur when the ranking of two items depends systematically on the method used to elicit it [12]. The classic cases of preference reversal are related to decisions involving pairs of simple monetary gambles. In each of these pairs, one bet (the 'P-bet') offers a relatively large chance of a modest prize, while the other (the '$-bet') offers a smaller chance of a larger prize. In a typical preference reversal experiment, a given subject makes a straight choice between the two bets and also states a monetary valuation for each of them. The classic finding is a puzzling tendency for subjects to choose the P-bet over the $-bet in the choice task but to place a strictly higher monetary value on the $-bet. Consider for example the following experiment to illustrate this phenomenon [47]: individuals under suitable laboratory conditions are asked if they prefer lottery A to lottery B. In lottery A a random dart is thrown to the interior of a circle. If it hits a straight line drawn from the center of the circle to the circumference, the subject is paid \$0 and if it hits anywhere else, the subject is paid \$4. There is a very high probability of winning, so this lottery is called the P-bet. If lottery B is chosen, a random dart is thrown to the interior of the circle and the subject receives either \$16 or \$0 depending upon where the dart hits (see Figure 1.1). Psychologists have observed that a large proportion of people will indicate a preference for lottery A but place a higher value on lottery B. The preference measured one way is the reverse of the preference measured another and seemingly theoretically compatible way.

In a final note, we would like to give an experimental example that shows the existence of situations in which it is mathematically clear which alternative should be preferred, but in which many people still prefer the other alternative. Consider for example the concept of ratio bias [1]: when judging the probability of a low probability event, many people judge it as less likely when it is expressed as a ratio of small numbers (e.g. $1 : 10$) than of large numbers (e.g. $10 : 100$). Imagine for example two bags (A and B) with red and white balls. Bag A contains 10 balls, 1 of which is red, and bag B contains 100 balls, 10 of which are red. The ratio of red balls is therefore the same for both bags. Suppose participants to the experiment are offered a certain reward if when extracting at random a ball frome one of the bags, it turns out to be red. Most of the participants usually choose bag B because, as they correctly assert, it contains more red balls. This ratio bias is even present in situations where bag B offers a smaller probability of winning than bag A. In [34], the ratios were $1 : 10$ and $8 : 100$ and almost half of the participants preferred the latter option. The subjects reported that although they knew the odds were against them, they felt they had a better chance when there were more red beans (beans were used instead of balls). In this work, however, we never deal directly with experimental values of preference relations obtained by individuals and such surprising unlogical results will therefore not be encountered.



lottery A: P-bet lottery B: \$-bet

Figure 1.1: *A psychological experiment.*

1.2 Random variables and distribution functions

Most information contained in this section was taken from [64]. We begin by recalling the definition of a distribution function and joint distribution function.

DEFINITION – 1.2.1: *A distribution function is a function F with domain $\overline{\mathbb{R}}$ such that*

1. *F is nondecreasing,*
2. *$F(-\infty) = 0$ and $F(+\infty) = 1$.*

A joint distribution function is a function H with domain $\overline{\mathbb{R}}^2$ such that

1. *H is 2-increasing (see Definition 1.1.3),*
2. *$H(x, -\infty) = H(-\infty, y) = 0$, and $H(+\infty, +\infty) = 1$.*

Note that the above definition does not mention random variables. A distribution function of the random variable (r.v.) X is the function F such that for all $x \in \overline{\mathbb{R}}$, $F(x) = \mathrm{Prob}\{X \leq x\}$, where $\mathrm{Prob}\{X \leq x\}$ is the probability that the r.v. X takes a value less than or equal to x. Distribution functions of r.v. can be right-continuous, left-continuous or both. However, in practice they are usually assumed to be at least right-continuous. A r.v. is continuous if its distribution function is continuous. Sklar's Theorem gives the connection between copulas, introduced in Definition 1.1.3, and joint distribution functions of r.v. [69, 71].

THEOREM – 1.2.2: *Let X_1 and X_2 be random variables with distribution functions F_{X_1} and F_{X_2}, respectively, and joint distribution function $F_{X_1,X_2}(x_1, x_2)$. Then there exists a copula C such that for all $x_1, x_2 \in \overline{\mathbb{R}}$*

$$F_{X_1,X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)) . \tag{1.14}$$

If F_{X_1} and F_{X_2} are continuous, then C is unique; otherwise, C is uniquely determined on $\mathrm{Ran}F_{X_1} \times \mathrm{Ran}F_{X_2}$. Conversely, if C is a copula and F_{X_1} and F_{X_2} are distribution functions, then the function F_{X_1,X_2} defined by (1.14) is a joint distribution function with margins F_{X_1} and F_{X_2}.

When the r.v. X_1 and X_2 are coupled by the $T_{\mathbf{P}}$-copula, i.e. when it holds that $F_{X_1,X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$, they are independent. As was already mentioned, any copula C is located between the $T_{\mathbf{L}}$-copula and the $T_{\mathbf{M}}$-copula. Note that in the theory of copulas the notation W (resp. M) is used for $T_{\mathbf{L}}$ (resp. $T_{\mathbf{M}}$) and this bound is called the Fréchet-Hoeffding lower bound (resp. Fréchet-Hoeffding upper bound). In this work, we will maintain the triangular norm notations $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$.

A copula C induces a probability measure on $[0, 1]^2$ via $V_C([0, u] \times [0, v]) = C(u, v)$, with $u, v \in [0, 1]$ (the so-called C-measure). C-measures are often called doubly stochastic measures, since for any measurable subset S of $[0, 1]$, $V_C(S \times [0, 1]) = V_C([0, 1] \times S) = \lambda(S)$, where λ denotes an ordinary Lebesgue measure on $[0, 1]$. For any copula C, let

$$C(u, v) = A_C(u, v) + S_C(u, v),$$

where

$$A_C(u,v) = \int_0^u \int_0^v \frac{\partial^2}{\partial s \partial t} C(s,t)\,dt\,ds \text{ and } S_C(u,v) = C(u,v) - A_C(u,v)\,.$$

If $C \equiv A_C$ on $[0,1]^2$ — that is, if considered as a joint distribution function, C has a joint density given by $\partial^2 C(u,v)/\partial u \partial v$ — then C is absolutely continuous, whereas if $C \equiv S_C$ on $[0,1]^2$ — that is, if $\partial^2 C(u,v)/\partial u \partial v = 0$ almost everywhere in $[0,1]^2$ — then C is singular. It now holds that the $T_{\mathbf{P}}$-copula is absolutely continuous, while the extreme copulas $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$ are singular.

Chapters 4 and 6 of this work are concerned with comparing collections of m random variables. In these chapters, random variables will be pairwisely compared, where no restrictions are laid upon the 2-copulas used to couple each pair of r.v. It is therefore not required that the set of m random variables forms an m-dimensional random vector, or in other words, the 2-copulas binding the pairs of r.v. need not be compatible. On the contrary, the set of 3 r.v. which are all pairwisely coupled by $T_{\mathbf{L}}$ will be investigated in Chapter 6, while no 3-dimensional random vector exists for which all 2-dimensional marginal distributions are coupled by $T_{\mathbf{L}}$.

For a thorough introduction on copulas we refer to [64], and for a broader view on multivariate models we refer to [52].

1.3 Game-theoretic concepts

Chapters 5 and 7 deal with characterizing the optimal strategies in three game variants and we therefore introduce the needed game-theoretical concepts in this section. The games that will be discussed are so-called *non-cooperative games*. In these games the goal of each participant (player) is to achieve the largest possible individual profit (*payoff*). The process of the game consists of each one of the players choosing a certain strategy $s_i \in S_i$. Thus as a result of each "round" of the game, a system of strategies $(s_1, \ldots, s_n) = s$ is put together. This system is called a *situation*.

A situation s is *admissible* for a player if by replacing her present strategy in this situation with some other strategy, the player is unable to increase her payoff. A situation s, which is admissible for all the players is called an *equilibrium situation* (also called Nash equilibrium). An *equilibrium strategy* of a player in a non-cooperative game is a strategy that appears in at least one equilibrium situation of the game.

An *antagonistic game* is a game with two players only and the values of the payoff function for these players in each situation are the same in absolute value but of the opposite sign: $a^1_{ij} = -a^2_{ij}$, with a^1_{ij}, a^2_{ij} the payoff functions of the respective players when player 1 chooses strategy i and player 2 chooses strategy j. Antagonistic games in which each player possesses a finite number of strategies are called *matrix games*.

An equilibrium situation for the particular case of an antagonistic game is called a *saddle point* and the equilibrium strategies of the players are then

$$\frac{1}{72} \times \begin{pmatrix} 0 & -4 & -4 & -4 & -9 & -9 & -9 & -14 & -14 & -19 & -24 \\ 4 & 0 & 0 & 0 & -4 & -5 & -6 & -9 & -10 & -14 & -18 \\ 4 & 0 & 0 & 0 & -3 & -3 & -3 & -6 & -6 & -9 & -12 \\ 4 & 0 & 0 & 0 & -4 & -2 & 0 & -6 & -4 & -8 & -12 \\ 9 & 4 & 3 & 4 & 0 & -1 & -3 & -4 & -6 & -9 & -12 \\ 9 & 5 & 3 & 2 & 1 & 0 & 0 & -2 & -2 & -4 & -6 \\ 9 & 6 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 9 & 6 & 6 & 4 & 2 & 0 & 0 & -2 & -4 & -6 \\ 14 & 10 & 6 & 4 & 6 & 2 & 0 & 2 & 0 & 0 & 0 \\ 19 & 14 & 9 & 8 & 9 & 4 & 0 & 4 & 0 & 0 & 0 \\ 24 & 18 & 12 & 12 & 12 & 6 & 0 & 6 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1.2: *Payoff matrix for the* $(6, 12)$*-game.*

called their *optimal strategies*. It is noteworthy to mention the following two properties: firstly, the value of the payoff function in each saddle point is the same, and secondly, if (x_1, y_1) and (x_2, y_2) are two different saddle points, then so are (x_1, y_2) and (x_2, y_1).

A matrix game is completely determined by its *payoff matrix*, which is defined by the matrix $A = [a^1_{ij}]$. An example of a payoff matrix is given in Figure 1.2. As the value in the payoff matrix of a matrix game is the same for each saddle point, all equilibrium strategies are equivalent for the players, hence the term optimal strategy. A matrix game with a payoff matrix $A = [a^1_{ij}]$ for which $a^1_{ij} = -a^1_{ji}$, is called a *symmetric game*. The value in a saddle point of the payoff matrix of a symmetric game equals zero. This means that in a symmetric game, if both players are infinitely smart, the payoff of both players will be zero and they therefore neither win nor lose.

For a player to maximize her payoff, she needs to choose an optimal strategy, if there is one. If she chooses this strategy, she is assured that her payoff is greater or equal to 0, no matter which strategy the other player chooses. If on the other hand, she doesn't choose an optimal strategy, but the other player does, she is assured that her payoff is less or equal to 0. It is therefore best to choose an optimal strategy. In the payoff matrix of Figure 1.2, which corresponds to the $(6, 12)$-game (see Chapter 5), the saddle points have been encircled. As can be deduced from this matrix, there are 4 optimal strategies and therefore 16 saddle points. It can be verified that for each row containing a saddle point the payoffs are greater or equal to 0 and for each column containing a saddle point they are less or equal to zero. For any situation that isn't a saddle point there can either be found a situation on the same row that has a smaller payoff, or a situation in the same column that has a bigger payoff.

If there is no saddle point, it is natural that the players should seek in these cases additional strategic opportunities in order to assure for themselves the largest possible profit. It turns out that it is desirable that they choose their strategies for this purpose randomly, in a well-chosen way. We then speak of a *mixed strategy* instead of a *pure strategy* and of equilibrium situations in mixed strategies. It can be shown that every matrix game has at least one equilibrium situation in mixed strategies. These can usually be obtained numerically by using linear programming. The payoff in an equilibrium situation in mixed strategies is, for a symmetric game, also given by 0. Our attempts to find closed expressions for the equilibrium situations in mixed strategies for the three game variants have failed, but we did obtain the closed expressions for the optimal strategies, if there are any. Note that, when there are multiple optimal strategies, there are infinitely many equilibrium situations in mixed strategies: the probabilities can be distributed over all optimal strategies. In the remainder of this work, mixed strategies will not be encountered again.

For further reading on game theory, we refer to [81, 82] and the seminal work by von Neumann and Morgenstern [80].

1.4 The theory of partitions – basic concepts

Some parts of this work use some basic notions from partition theory. To that extent, we briefly introduce the concepts that are applicable, starting with the definition of a partition itself.

DEFINITION – 1.4.1: *A partition of a positive integer σ is a finite nondecreasing sequence of positive integers $(i_1, i_2, \ldots, i_n)$ such that $\sum_j i_j = \sigma$. The i_j are called the parts of the partition.*

Note that in partition theory the parts of a partition are usually ordered nonincreasingly, but for the purpose of this work it was decided to order them nondecreasingly. In this work, we are mainly concerned about a specific type of partition, called an (n, σ) partition.

DEFINITION – 1.4.2: *An (n, σ) partition is a nondecreasingly ordered list $(i_1, i_2, \ldots, i_n)$ of n strictly positive integers summing up to σ ($\sum_j i_j = \sigma$, $i_j \in \mathbb{N}_0$).*

Sometimes it is useful to use a notation that makes explicit the number of times that a particular integer occurs as a part and this notation is called the multiplicity representation.

DEFINITION – 1.4.3: *For a partition $\pi = (i_1, i_2, \ldots, i_n)$, the multiplicity representation is given by $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$, where exactly t_l of the i_j are equal to l. When $t_l = 0$, the term l^{t_l} can be omitted.*

For the multiplicity representation $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$ of a given (n, σ) partition π it clearly holds that $0 \leq t_i \leq n$, $\sum_{i>0} t_i = n$ and $\sum_{i>0} i\, t_i = \sigma$.

In partition theory everything usually comes down to a counting problem. The most basic problem is, of course, counting the number of partitions of a number.

DEFINITION – 1.4.4: *The partition function $p(n)$ denotes the number of partitions of n.*

Obviously, $p(n) = 0$ when n is negative. By definition, we set $p(0) = 1$ with the observation that the empty sequence forms the only partition of zero. However basic this counting problem might sound, it is in fact a very difficult problem which has been solved only as late as in the beginning of the twentieth century, by some of the greatest mathematical minds of their time. Conceptually, counting the number of partitions is very similar to counting the number of compositions of an integer. Compositions are merely partitions in which the order of the summands is considered. For example, there are seven partitions of 5: (5), $(1,4)$, $(2,3)$, $(1,1,3)$, $(1,2,2)$, $(1,1,1,2)$, $(1,1,1,1,1)$. There are 16 compositions of 5: (5), $(1,4)$, $(4,1)$, $(2,3)$, $(3,2)$, $(1,1,3)$, $(1,3,1)$, $(3,1,1)$, $(1,1,1,2)$, $(1,1,2,1)$, $(1,2,1,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$, $(1,2,2)$, $(2,1,2)$ and $(2,2,1)$. The formula for the number of compositions of a positive integer n is simply given by 2^{n-1}. On the contrary, finding a closed form for the function $p(n)$ has proven to be a very difficult task, solvable only by people gifted with great mathematical genius. This magnificent feat was accomplished mostly by G. H. Hardy and S. Ramanujan while fully perfected by H. Rademacher. A historical overview of this work is given in [3], however we do not want to leave the reader without having seen the identity of the Hardy-Ramanujan-Rademacher expansion of $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{x-\frac{1}{24}}} \right]_{x=n},$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ d(h,k)=1}} \omega_{h,k} \mathrm{e}^{-2\pi i n h/k}$$

with $\omega_{h,k}$ given by

$$\omega_{h,k} = \exp(\pi i s(h,k)),$$

where $s(h,k)$ is the Dedekind sum:

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \left\lfloor\frac{\mu}{k}\right\rfloor - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \left\lfloor\frac{h\mu}{k}\right\rfloor - \frac{1}{2}\right).$$

It may please the reader to know that by having just read the above identity, she has successfully read through the hardest to comprehend theorem stated in this work, which isn't merely because the proof has been excluded.

For the modest purposes of this work, we need to introduce two additional concepts from partition theory. Sometimes it is better to graphically represent partitions. Therefore, to each partition π is associated its graphical representation $\mathscr{G}_\pi$ (also known as Ferrers graph), which formally is the set of points with integral coordinates (i,j) in the plane such that if $\pi = (i_1, i_2, \ldots, i_n)$, then $(i,j) \in \mathscr{G}_\pi$ if and only if $0 \geq i \geq -n+1$, $0 \leq j \leq i_{|i|+1} - 1$ [3]. As is so often the case, an example will set the mind: Figure 1.3 shows the Ferrers graph corresponding to the partition $2+3+4+4+7$. Note that the (nondecreasing) alternative way of representing the graph is obtained by putting the graph from the figure upside down. In the counting problems encountered in this

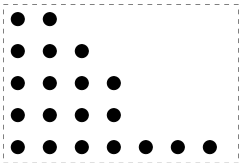


Figure 1.3: *A Ferrers graph.*

work, the concept of restricted partitions will play the central role. Before introducing these special types of partition problems, we need the concept of infinite product generating functions.

DEFINITION – 1.4.5: *The generating function $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series $f(q) = \sum_{n\geq 0} a_n q^n$.*

The existence of a generating function for a given counting problem is of great value, as the counting is then reduced from actually constructing all satisfying cases to a simple mathematical expansion which can be done by any good numerical software tool.

DEFINITION – 1.4.6: *The value $p(N, M, n)$ denotes the number of partitions of n into at most M parts, each $\leq N$. These partitions are called restricted partitions.*

Clearly, $p(N, M, n) = 0$ when $n > MN$, while $p(N, M, NM) = 1$. Therefore the generating function

$$G(N, M; q) = \sum_{n\geq 0} p(N, M, n) q^n$$

is a polynomial in q of degree NM. We can now end by stating the following important theorem, which provides the generating function for restricted partitions.

THEOREM – 1.4.7: *For $M, N \geq 0$,*

$$\begin{aligned} G(N, M; q) &= \frac{(1-q^{N+M})(1-q^{N+M-1})\ldots(1-q^{M+1})}{(1-q^N)(1-q^{N-1})\ldots(1-q)} \\ &= \frac{(q)_{N+M}}{(q)_N (q)_M}\,. \end{aligned} \tag{1.15}$$

Thanks to the above theorem which provides the generating function for $p(N, M, n)$, if we can reformulate the counting problems we will encounter in later chapters so that the number is a function of only $p(N, M, n)$, we have the full mathematical characterization of these values.

The polynomials (1.15) are known as the Gaussian polynomials, defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{, if } 0 \leq m \leq n\,, \\ 0 & \text{, otherwise}\,. \end{cases}$$

Note that

$$\begin{bmatrix} N+M \\ M \end{bmatrix} = G(N, M; q)\,.$$

All statements are true in some sense, false in some sense, meaningless in some sense, true and false in some sense, true and meaningless in some sense, false and meaningless in some sense, and true and false and meaningless in some sense.
— PUBLIC SERVICE CLARIFICATION BY THE SRI SYADASTI SCHOOL OF SPIRITUAL WISDOM, WILMETTE

I hate definitions.
— BENJAMIN DISRAELI

It is a mistake to think
you can solve any major problem
just with potatoes.
— DOUGLAS N. ADAMS

2 Cycle-transitivity

Throughout this work, the concept of cycle-transitivity will pop up as being ideally suited as a descriptive tool for probabilistic relations. Cycle-transitivity is a very general framework that enables representing diverse already known types of transitivity. It has also turned out to be ideally suited for describing new types of transitivity that are encountered when comparing random variables, as will become clear in subsequent chapters. Finally, the use of upper bound functions for describing the transitivity is a convenient way to show relationships between different types of transitivity. In this chapter, we will develop this general framework [16, 19, 17]. The key feature is the expression of transitivity using a cyclic evaluation of the values of the probabilistic relation: triangles (i.e. any three points, corresponding to three alternatives) are visited in a cyclic manner. An upper bound function acting upon the ordered weights encountered provides an upper bound for the "sum minus 1" of these weights. Commutative quasi-copulas allow to translate a general definition of fuzzy transitivity (when applied to probabilistic relations) elegantly into the framework of cycle-transitivity. Similarly, a general notion of stochastic transitivity corresponds to a particular class of upper bound functions.

Comparison functions and probabilistic relations are a convenient tool for expressing the result of the pairwise comparison of a set of alternatives [15] and appear in various fields such as game theory [36], voting theory [44, 65] and psychological studies on preference and discrimination in (individual or collective) decision-making methods [35]. Probabilistic relations are particularly popular in fuzzy set theory where they are used for representing intensities of preference [11, 54, 78]. In group decision making, probabilistic relations represent collective preferences and are built from individual preferences, either by aggregation methods [42] or consensus-reaching processes [54]. In social choice theory, there is a vast literature on the study of choice rules [13, 53, 65] (resp. choice correspondences [36, 58]) given preferences expressed in terms of probabilistic relations (resp. comparison functions). The remaining chapters in this work will all involve such probabilistic relations, which is why we lay an emphasis on them in the present chapter.

Although T-transitivity has been devised for fuzzy relations, which aren't necessarily probabilistic relations, we begin Section 2.1 with a careful study of $T_{\mathbf{P}}$-transitivity for probabilistic relations. Our observations will motivate the introduction of the concept of cycle-transitivity. Particular attention will be paid to so-called self-dual upper bound functions. In Section 2.3, we show how fuzzy transitivity, and in particular T-transitivity, fits into the new framework. Commutative quasi-copulas, and in particular members of the Frank t-norm family, permit an elegant reformulation. In Section 2.4, we propose a broad definition of stochastic transitivity, of which strong, moderate and weak stochastic transitivity are well-known instances. It is shown under which conditions this type of transitivity can be cast into the cycle-transitivity framework as well. Also, the discussion of self-duality leads to remarkable results, attributing a particular role to t-conorms. Finally, Section 2.5 briefly considers a different definition of cycle-transitivity based on a so-called symmetric payoff relation involving a simple transformation of the values of the probabilistic relation.

2.1 Preliminaries

2.1.1 Notations

Consider an arbitrary universe A. For a probabilistic relation Q on A, we write $q_{ab} := Q(a,b)$. For any $(a,b,c) \in A^3$, let

$$\begin{aligned} \alpha_{abc} &= \min(q_{ab}, q_{bc}, q_{ca})\,, \\ \beta_{abc} &= \mathrm{median}(q_{ab}, q_{bc}, q_{ca})\,, \\ \gamma_{abc} &= \max(q_{ab}, q_{bc}, q_{ca})\,. \end{aligned} \tag{2.1}$$

Note that A need not be countable, which is why we use the indices abc instead of ijk. It now obviously holds that

$$\alpha_{abc} \leq \beta_{abc} \leq \gamma_{abc}\,, \tag{2.2}$$

and also

$$\alpha_{abc} = \alpha_{bca} = \alpha_{cab}\,, \quad \beta_{abc} = \beta_{bca} = \beta_{cab}\,, \quad \gamma_{abc} = \gamma_{bca} = \gamma_{cab}\,. \tag{2.3}$$

On the other hand, the probabilistic nature of Q implies that

$$\alpha_{cba} = 1 - \gamma_{abc}\,, \quad \beta_{cba} = 1 - \beta_{abc}\,, \quad \gamma_{cba} = 1 - \alpha_{abc}\,. \tag{2.4}$$

2.1.2 $T_{\mathbf{P}}$-transitivity

To point out a possible way of generalizing T-transitivity (for probabilistic relations), we consider $T_{\mathbf{P}}$-transitivity for a probabilistic relation Q on A. For any $a, b, c \in A$, there are six conditions to be satisfied, namely

$$\begin{aligned} q_{ac}\, q_{cb} &\leq q_{ab}\,, \quad q_{ba}\, q_{ac} \leq q_{bc}\,, \quad q_{cb}\, q_{ba} \leq q_{ca}\,, \\ q_{bc}\, q_{ca} &\leq q_{ba}\,, \quad q_{ca}\, q_{ab} \leq q_{cb}\,, \quad q_{ab}\, q_{bc} \leq q_{ac}\,. \end{aligned}$$

Since Q is reciprocal, these conditions can be expressed in terms of α_{abc}, β_{abc} and γ_{abc} solely, as follows

$$\begin{aligned} (1-\beta_{abc})(1-\gamma_{abc}) &\leq \alpha_{abc}\,, \quad \beta_{abc}\,\gamma_{abc} \leq 1-\alpha_{abc}\,, \\ (1-\alpha_{abc})(1-\gamma_{abc}) &\leq \beta_{abc}\,, \quad \alpha_{abc}\,\gamma_{abc} \leq 1-\beta_{abc}\,, \\ (1-\alpha_{abc})(1-\beta_{abc}) &\leq \gamma_{abc}\,, \quad \alpha_{abc}\,\beta_{abc} \leq 1-\gamma_{abc}\,. \end{aligned} \tag{2.5}$$

The three left-hand inequalities of (2.5) can be rewritten as

$$\begin{aligned} \beta_{abc}\gamma_{abc} &\leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,, \\ \alpha_{abc}\gamma_{abc} &\leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,, \\ \alpha_{abc}\beta_{abc} &\leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,. \end{aligned}$$

From (2.2) it follows that $\alpha_{abc}\,\beta_{abc} \leq \alpha_{abc}\,\gamma_{abc} \leq \beta_{abc}\,\gamma_{abc}$. Therefore only the first inequality should be withheld as a condition for $T_{\mathbf{P}}$-transitivity. Similarly, the three right-hand inequalities of (2.5) can be rewritten as

$$\begin{aligned}
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1 - \beta_{abc})(1 - \gamma_{abc})\,, \\
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1 - \alpha_{abc})(1 - \gamma_{abc})\,, \\
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1 - \alpha_{abc})(1 - \beta_{abc})\,.
\end{aligned}$$

From (2.2) it now follows that only the last inequality should be retained. The six inequalities (2.5) are therefore equivalent to the double inequality

$$\beta_{abc}\,\gamma_{abc} \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq 1 - (1 - \alpha_{abc})(1 - \beta_{abc})\,. \tag{2.6}$$

The way we arrived at this double inequality immediately shows that if it holds for $(a, b, c) \in A^3$, then it also holds for all permutations of (a, b, c). A direct proof of this claim, however, provides us with some further insights. Let us denote the upper and lower bounds in (2.6) as $u(\alpha_{abc}, \beta_{abc})$ and $l(\beta_{abc}, \gamma_{abc})$, respectively. We observe the following type of duality:

$$l(\beta_{abc}, \gamma_{abc}) = 1 - u(1 - \gamma_{abc}, 1 - \beta_{abc})\,. \tag{2.7}$$

Suppose (2.6) holds for (a, b, c), then (2.4) and (2.7) lead to

$$\begin{aligned}
\alpha_{cba} + \beta_{cba} + \gamma_{cba} - 1 &= 1 - (\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1) \\
&\geq 1 - u(\alpha_{abc}, \beta_{abc}) \\
&= 1 - u(1 - \gamma_{cba}, 1 - \beta_{cba}) = l(\beta_{cba}, \gamma_{cba})\,.
\end{aligned}$$

Similarly, we obtain $\alpha_{cba} + \beta_{cba} + \gamma_{cba} - 1 \leq u(\alpha_{cba}, \beta_{cba})$. Hence, (2.6) also holds for (c, b, a).

2.2 The definition of cycle-transitivity

The simple formulation (2.6)–(2.7) of $T_{\mathbf{P}}$-transitivity for probabilistic relations has been the source of inspiration for a new way of describing the transitivity of probabilistic relations. Let us denote $\Delta = \{(x, y, z) \in [0, 1]^3 \mid x \leq y \leq z\}$ and consider a function $U : \Delta \to \mathbb{R}$, then, in analogy to (2.6), we could call a probabilistic relation Q on A transitive w.r.t. U if for any $a, b, c \in A$ it holds that

$$1 - U(1 - \gamma_{abc}, 1 - \beta_{abc}, 1 - \alpha_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})\,.$$

In case of $T_{\mathbf{P}}$-transitivity, the corresponding function $U_{\mathbf{P}}$ is given by

$$U_{\mathbf{P}}(\alpha, \beta, \gamma) = 1 - (1 - \alpha)(1 - \beta) = \alpha + \beta - \alpha\beta\,. \tag{2.8}$$

The minimal requirement we will impose is that the probabilistic representation Q of any transitive complete relation R given in (1.8) satisfies any form of cycle-transitivity. To that end, U should satisfy the following conditions:

$$\begin{aligned}
U(0, 1/2, 1) &\geq 1/2\,, \qquad & U(1/2, 1/2, 1/2) &\geq 1/2\,, \\
U(0, 0, 1) &\geq 0\,, & U(0, 1, 1) &\geq 1\,.
\end{aligned} \tag{2.9}$$

These conditions are for instance satisfied for any $U \geq$ median. The choice of requiring the above conditions does have an impact, however. These conditions imply that the constant function $U_{\mathrm{A}}(\alpha, \beta, \gamma) = 1/2$ is not an upper bound function which implies that additive transitivity, defined in (1.12), cannot be represented in the cycle-transitivity framework. This is, however, the only relevant type of transitivity we have found that does not fit into the framework, which makes additive transitivity quite unique.

To explain where the above four conditions come from, we first note that for any transitive probabilistic relation given in (1.8) with three elements $\{a, b, c\}$ it holds that $(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$ is either $(1/2, 1/2, 1/2)$, either $(0, 1/2, 1)$, either $(0, 1, 1)$ or $(0, 0, 1)$. Therefore the following inequalities must hold, which are equivalent to (2.9).

$$\begin{gathered} L(1/2,1/2,1/2) \leq 1/2 + 1/2 + 1/2 - 1 \leq U(1/2,1/2,1/2)\,, \\ L(0,1/2,1) \leq 0 + 1/2 + 1 - 1 \leq U(0,1/2,1)\,, \\ L(0,1,1) \leq 0 + 1 + 1 - 1 \leq U(0,1,1)\,, \\ L(0,0,1) \leq 0 + 0 + 1 - 1 \leq U(0,0,1)\,. \end{gathered}$$

In a similar fashion, a requirement could be to insist that the only $\{0, 1/2, 1\}$-valued probabilistic relations that are cycle-transitive w.r.t. U are the probabilistic representations of transitive complete relations. For this requirement to be satisfied, the following additional conditions would have to be fulfilled:

$$\begin{gathered} U(0,0,0) < -1 \text{ or } U(1,1,1) < 2\,, \\ U(0,0,1/2) < -1/2 \text{ or } U(1/2,1,1) < 3/2\,, \\ U(0,1/2,1/2) < 0 \text{ or } U(1/2,1/2,1) < 1\,. \end{gathered} \tag{2.10}$$

The above strict inequalities are obtained in an analogous way as the inequalities (2.9). As this requirement would seriously limit the generality of the framework, it is not imposed. Dice-transitivity, which we will encounter in subsequent chapters, for instance, does not satisfy the above three conditions. A final restriction made to the class of upper bounds is that for any such upper bound the corresponding lower bound should not exceed it, in any point taken from Δ.

DEFINITION – 2.2.1: *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U(0,0,1) \geq 0$ *and* $U(0,1,1) \geq 1$*;*

(ii) for any $(\alpha, \beta, \gamma) \in \Delta$*:*

$$U(\alpha, \beta, \gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geq 1\,. \tag{2.11}$$

The class of upper bound functions is denoted $\mathcal{U}$.

Note that the definition of an upper bound function does not include any monotonicity condition. The function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha, \beta, \gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha) \tag{2.12}$$

is called the *dual lower bound function* of a given upper bound function U. Inequality (2.11) then simply expresses that $L \leq U$. Note that the conditions $U(0,1/2,1) \geq 1/2$ and $U(1/2,1/2,1/2) \geq 1/2$ follow from (2.11) and are therefore omitted in the above definition. One easily verifies that $U_{\mathbf{P}}$ belongs to $\mathcal{U}$ and, moreover, also satisfies (2.10).

DEFINITION – 2.2.2: *A probabilistic relation Q on A is called cycle-transitive w.r.t. an upper bound function $U \in \mathcal{U}$ if for any $(a,b,c) \in A^3$ it holds that*

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}), \tag{2.13}$$

where L is the dual lower bound function of U.

Using the above terminology, the results of Section 2.1 can be rephrased as follows: a probabilistic relation Q is $T_{\mathbf{P}}$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U_{\mathbf{P}}$, defined in (2.8). In general, due to the built-in duality, it holds that if (2.13) is true for some (a,b,c), then this is also the case for any permutation of (a,b,c). In practice, it is therefore sufficient to check (2.13) for a single permutation of any $(a,b,c) \in A^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in A^3$ that aren't cyclic permutations of one another, e.g. (a,b,c) and (c,b,a).

PROPOSITION – 2.2.3: *A probabilistic relation Q on A is cycle-transitive w.r.t. an upper bound function $U \in \mathcal{U}$ if for any $(a,b,c) \in A^3$ it holds that*

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}). \tag{2.14}$$

Proof:
We need to show that, given the inequality

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}),$$

the inequalities

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1$$

and

$$\alpha_{cba} + \beta_{cba} + \gamma_{cba} - 1 \leq U(\alpha_{cba}, \beta_{cba}, \gamma_{cba}),$$

are equivalent. As $\gamma_{cba} = 1 - \alpha_{abc}$, $\beta_{cba} = 1 - \beta_{abc}$ and $\alpha_{cba} = 1 - \gamma_{abc}$, the equivalence follows directly from the definition of the dual lower bound. ∎

Note that the upper bound function $U(\alpha, \beta, \gamma) = 2$ will often be used to express that for the given values there is no restriction at all (indeed, $\alpha + \beta + \gamma - 1$ is always bounded from above by 2). For two upper bound functions such that $U_1 \leq U_2$, it clearly holds that cycle-transitivity w.r.t. U_1 implies cycle-transitivity w.r.t. U_2. However, it is clear that $U_1 \leq U_2$ is not a necessary condition for the latter implication to hold. To give an example that we will encounter later on, cycle-transitivity w.r.t. the upper bound function $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$ implies cycle-transitivity w.r.t. the upper bound function $U_{\mathbf{L}}(\alpha, \beta, \gamma) = \min(\alpha + \beta, 1)$, while $1 \geq \min(\alpha + \beta, 1)$ and $U'_{\mathbf{L}} \neq U_{\mathbf{L}}$.

Two upper bound functions U_1 and U_2 will be called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that

$$\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$$

is equivalent to

$$\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma)\,.$$

Suppose, for instance, that the inequality $\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$ can be rewritten as

$$\alpha \leq h(\beta, \gamma)\,,$$

then an equivalent upper bound function U_2 is given by

$$U_2(\alpha, \beta, \gamma) = \beta + \gamma - 1 + h(\beta, \gamma)\,.$$

In this way, it is often possible to reduce an upper bound function in three variables to an equivalent upper bound function in only two of the variables α, β and γ. Another method of obtaining equivalent upper bound functions is as follows. For any $\mu > 0$, the inequality

$$\alpha + \beta + \gamma - 1 \leq U(\alpha, \beta, \gamma)$$

is clearly equivalent to

$$\alpha + \beta + \gamma - 1 \leq \frac{U(\alpha, \beta, \gamma) - (1 - \mu)(\alpha + \beta + \gamma - 1)}{\mu}\,.$$

Hence, cycle-transitivity w.r.t. the upper bound function U is equivalent to cycle-transitivity w.r.t. U_μ defined by

$$U_\mu(\alpha, \beta, \gamma) = \frac{U(\alpha, \beta, \gamma) - (1 - \mu)(\alpha + \beta + \gamma - 1)}{\mu}\,. \tag{2.15}$$

One easily verifies that if $U \in \mathcal{U}$ then $U_\mu \in \mathcal{U}$. Note that also the additional conditions (2.10) are preserved, under the above transformation.

2.2.1 Self-dual upper bound functions

If it happens that in (2.11) the equality holds for all $(\alpha, \beta, \gamma) \in \Delta$, i.e.

$$U(\alpha, \beta, \gamma) + U(1 - \gamma, 1 - \beta, 1 - \alpha) = 1\,, \tag{2.16}$$

then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, then also (2.13) and (2.14) can only hold with equality. Furthermore, it holds that $U(0, 0, 1) = 0$

and $U(0,1,1) = 1$. Note that if U is self-dual, then also any upper bound function U_μ defined in (2.15) is self-dual.

The simplest self-dual upper bound function is given by the median, i.e. $U_{\mathbf{M}}(\alpha,\beta,\gamma) = \beta$, and further on we will prove that this is precisely the upper bound function corresponding to $T_{\mathbf{M}}$-transitivity of probabilistic relations, when it is reformulated in the framework of cycle-transitivity.

Another example of a self-dual upper bound function is the function U_E defined by

$$U_E(\alpha,\beta,\gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma\,. \tag{2.17}$$

Cycle-transitivity w.r.t. U_E of a probabilistic relation Q on A can also be expressed as

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 = \alpha_{abc}\beta_{abc} + \alpha_{abc}\gamma_{abc} + \beta_{abc}\gamma_{abc} - 2\alpha_{abc}\beta_{abc}\gamma_{abc}\,,$$

or, equivalently, as:

$$\alpha_{abc}\beta_{abc}\gamma_{abc} = (1-\alpha_{abc})(1-\beta_{abc})(1-\gamma_{abc})\,.$$

It is then easy to see that cycle-transitivity w.r.t. U_E is equivalent to the concept of multiplicative transitivity, which was already defined in (1.13). Note that the cycle-transitive version is more appropriate, although less intuitive, as it avoids division by zero.

It is possible to characterize the subfamily of self-dual upper bound functions that are polynomial functions. Indeed, introducing the new variables $\alpha' = \alpha - 1/2$, $\beta' = \beta - 1/2$, $\gamma' = \gamma - 1/2$, and the new function U' defined by

$$U'(\alpha',\beta',\gamma') = U(\alpha'+1/2, \beta'+1/2, \gamma'+1/2) - 1/2\,, \tag{2.18}$$

the self-duality of U becomes equivalent to

$$U'(\alpha',\beta',\gamma') = -U'(-\gamma',-\beta',-\alpha')\,, \tag{2.19}$$

which should hold for all $-1/2 \le \alpha' \le \beta' \le \gamma' \le 1/2$. We now give explicitly the polynomial solutions of the latter functional equation.

PROPOSITION – 2.2.4: *All polynomial solutions of* (2.19) *are given by*

$$U'(\alpha',\beta',\gamma') = \sum_{(i,j,k)\in\mathbb{N}^3} c_{ijk}(\alpha'\gamma')^i\beta'^j\left[\alpha'^k + (-1)^{j+k+1}\gamma'^k\right]\,, \tag{2.20}$$

where the c_{ijk} are arbitrary reals. For the corresponding function U, derived from (2.19)*, to be an upper bound function, $U'(-1/2,1/2,1/2) = 1/2$ must hold.*

Proof:
Suppose we have a polynomial solution $U'(\alpha',\beta',\gamma')$ of (2.19). As it is a polynomial function, we can always write the function in the following form:

$$U'(\alpha',\beta',\gamma') = \sum_{l=(l_1,l_2,l_3)\in\mathbb{N}^3} \delta_l\,\alpha'^{l_1}\beta'^{l_2}\gamma'^{l_3}\,,\text{with } \delta_l\in\mathbb{R}\,. \tag{2.21}$$

The condition (2.19) is equivalent to the following equality that must be satisfied for any $l = (l_1, l_2, l_3) \in \mathbb{N}^3$ and corresponding $l' = (l_3, l_2, l_1)$:

$$\begin{aligned}\delta_l\, \alpha'^{l_1} \beta'^{l_2} \gamma'^{l_3} + \delta_{l'}\, \alpha'^{l_3} \beta'^{l_2} \gamma'^{l_1} \;=\; -\Big(&\delta_l\, (-\gamma')^{l_1} (-\beta')^{l_2} (-\alpha')^{l_3} + \\ &\delta_{l'}\, (-\gamma')^{l_3} (-\beta')^{l_2} (-\alpha')^{l_1}\Big)\,.\end{aligned} \tag{2.22}$$

When the sum $l_1 + l_2 + l_3$ is odd, at least one of these conditions must be satisfied: $\delta_l = \delta_{l'}$ or $l_1 = l_3$. This corresponds to $k > 0$ or $k = 0$ in (2.20), and the corresponding term for the left-hand side of (2.22) in (2.20) is then given using the values

$$i = \min(l_1, l_3),\; j = l_2,\; k = \max(l_1, l_3) - i,\; c_{ijk} = \frac{\delta_l + \delta_{l'}}{2}\,.$$

When $l_1 + l_2 + l_3$ is even, it must hold that $\delta_l = -\delta_{l'}$, and we use the values

$$i = \min(l_1, l_3),\; j = l_2,\; k = \max(l_1, l_3) - i,\; c_{ijk} = \pm\delta_l\,,$$

where $c_{ijk} = \delta_l$ when $l_1 \geq l_3$ and $c_{ijk} = -\delta_l$ when $l_3 > l_1$. All terms from (2.21) can therefore be written in the form (2.20). Conversely, the functions U' defined in (2.20) satisfy (2.19) and can of course be written using the form (2.21). Therefore, when the equality (2.22) holds, the forms (2.20) and (2.21) produce the same set of functions, namely the polynomials and infinite series defined over Δ and satisfying (2.19).

It still needs to be verified, however, that the functions U derived from U' using (2.18) and satisfying (2.20), also satisfy the conditions mentioned in Definition 2.2.1. As U is self-dual, the conditions $U(0,1,1) \geq 1$ and $U(0,0,1) \geq 1$ can be combined into one condition: $U(0,1,1) = 1$, which is equivalent to $U'(-1/2, 1/2, 1/2) = 1/2$. ∎

As was already mentioned, (2.20) not only defines all self-dual polynomial functions, but also all self-dual infinite series. Returning to the original variables, all self-dual upper bound polynomial functions (or infinite series) U are given by

$$\begin{aligned}U(\alpha,\beta,\gamma) = \tfrac{1}{2} + \sum_{(i,j,k)\in\mathbb{N}^3} &c_{ijk}(\alpha - 1/2)^i\,(\beta - 1/2)^j\,(\gamma - 1/2)^i \times \\ &\Big[(\alpha - 1/2)^k + (-1)^{j+k+1}(\gamma - 1/2)^k\Big]\,,\end{aligned} \tag{2.23}$$

where the coefficients c_{ijk} are restricted in order to ensure that $U(0,1,1) = 1$ (and, equivalently, $U(0,0,1) = 0$).

By setting $c_{010} = 1/2$ and all other c_{ijk} to zero in (2.23), the self-dual upper bound function $U_{\mathbf{M}}$ is retrieved, while choosing $c_{110} = -1$, $c_{010} = 1/4$, $c_{001} = 1/2$ and all other $c_{ijk} = 0$, leads to the self-dual upper bound function U_E, corresponding to multiplicative transitivity.

2.3 Fuzzy transitivity as cycle-transitivity

2.3.1 Fuzzy transitivity

In this section, we reconsider the notion of T-transitivity. Instead of t-norms, we consider the more general class of conjunctors.

DEFINITION – 2.3.1: *A binary operation $f : [0,1]^2 \to [0,1]$ is called a conjunctor if it has the following properties:*

(i) Its restriction to $\{0,1\}^2$ coincides with the Boolean conjunction.

(ii) Monotonicity: f is increasing in each variable.

The following definition generalizes Definition 1.1.4.

DEFINITION – 2.3.2: *Let f be a conjunctor. A fuzzy relation R on A is called f-transitive if for any $(a,b,c) \in A^3$ it holds that*

$$f(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{2.24}$$

Typical examples of conjunctors are binary operations on $[0,1]$ that satisfy (ii) and have 1 as neutral element, i.e. $f(x,1) = f(1,x) = x$ for any $x \in [0,1]$. Such conjunctors are bounded from above by $T_{\mathbf{M}}$, i.e. $f(x,y) \leq \min(x,y)$, and have 0 as absorbing element, i.e. $f(x,0) = f(0,x) = 0$, for any $x \in [0,1]$.

In this work, we are mainly interested in two particular classes of commutative conjunctors with neutral element 1: the class of t-norms and the class of (commutative) (quasi-)copulas, both defined in Section 1.1.2 and finding their origin in the study of probabilistic metric spaces [70].

2.3.2 Fuzzy transitivity as cycle-transitivity

Although fuzzy transitivity was introduced for fuzzy relations, which are not necessarily reciprocal, we will only consider those that are, as cycle-transitivity has been designed specifically for these relations. A first immediate observation is the following proposition.

PROPOSITION – 2.3.3: *Let f be a commutative conjunctor such that $f \leq T_{\mathbf{M}}$. A probabilistic relation Q on A is f-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_f defined by*

$$U_f(\alpha,\beta,\gamma) = \min(\alpha+\beta-f(\alpha,\beta), \beta+\gamma-f(\beta,\gamma), \gamma+\alpha-f(\gamma,\alpha))\,. \tag{2.25}$$

Proof:
First of all, a simple verification shows that for any conjunctor f the property $f \leq T_{\mathbf{M}}$ guarantees that the function U_f defined in (2.25) belongs to $\mathcal{U}$. Indeed, as it then holds that $f(0,x) = 0$ we obtain $U_f(0,0,1) = \min(0,1,1) = 0$ and $U_f(0,1,1) = \min(1,1,1) = 1$. It also directly follows that $U_f(\alpha,\beta,\gamma) + U_f(1-\gamma, 1-\beta, 1-\alpha) = \min(\alpha+\beta-f(\alpha,\beta), \beta+\gamma-f(\beta,\gamma), \gamma+\alpha-f(\gamma,\alpha)) +$

$\min(2-\beta-\gamma-f(1-\gamma,1-\beta), 2-\alpha-\beta-f(1-\beta,1-\alpha), 2-\alpha-\gamma-f(1-\alpha,1-\gamma)) \geq \beta+1-\beta = 1$.

Consider a probabilistic relation Q on A and $(a,b,c) \in A^3$. Assume e.g. that $q_{ab} = \alpha_{abc}$, $q_{bc} = \beta_{abc}$ and $q_{ca} = \gamma_{abc}$. The six inequalities of type (2.24), guaranteeing f-transitivity, can be brought, by adding appropriate terms to both sides of the inequalities, into the following form (also omitting the indices abc):

$$\begin{aligned} f(1-\gamma,1-\beta)+\gamma+\beta-1 &\leq \alpha+\beta+\gamma-1\,,\\ f(1-\alpha,1-\gamma)+\alpha+\gamma-1 &\leq \alpha+\beta+\gamma-1\,,\\ f(1-\beta,1-\alpha)+\beta+\alpha-1 &\leq \alpha+\beta+\gamma-1\,,\\ \alpha+\beta+\gamma-1 &\leq -f(\beta,\gamma)+\beta+\gamma\,,\\ \alpha+\beta+\gamma-1 &\leq -f(\gamma,\alpha)+\gamma+\alpha\,,\\ \alpha+\beta+\gamma-1 &\leq -f(\alpha,\beta)+\alpha+\beta\,. \end{aligned}$$

Similarly as for $T_{\mathbf{P}}$-transitivity, these six inequalities are equivalent to the double inequality

$$L_f(\alpha,\beta,\gamma) \leq \alpha+\beta+\gamma-1 \leq U_f(\alpha,\beta,\gamma)\,,$$

with U_f given by (2.25) and L_f the dual lower bound function defined by (2.12): $1-U_f(1-\gamma,1-\beta,1-\alpha) = \max(\alpha+\beta+f(1-\beta,1-\alpha), \beta+\gamma+f(1-\gamma,1-\beta), \alpha+\gamma+f(1-\alpha,1-\gamma))-1$. Due to the commutativity of f, any other case, such as $q_{ab} = \alpha_{abc}$, $q_{bc} = \gamma_{abc}$ and $q_{ca} = \beta_{abc}$, leads to the same result. ∎

Note that in general the additional conditions (2.10) are not satisfied by an upper bound function of type (2.25). Due to the third condition from (2.10) this is only the case when $f(1/2,1/2) > 0$, a condition that is e.g. not fulfilled for $f = T_{\mathbf{L}}$.

2.3.3 The case of commutative quasi-copulas and copulas

Proposition 2.3.3 does not sufficiently emphasize the relevance of the concept of cycle-transitivity. It would be interesting to establish sufficient conditions bringing the upper bound function U_f in a simpler form, in analogy to the result obtained for $T_{\mathbf{P}}$.

PROPOSITION – 2.3.4: *Let f be a commutative conjunctor such that $f \leq T_{\mathbf{M}}$. If f is 1-Lipschitz, then a probabilistic relation Q on A is f-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_f defined by*

$$U_f(\alpha,\beta,\gamma) = \alpha+\beta-f(\alpha,\beta)\,. \tag{2.26}$$

Proof:
First, we observe that due to the monotonicity and commutativity of f, the 1-Lipschitz property of f can be stated equivalently as

$$y-f(x,y) \leq z-f(x,z)\,, \tag{2.27}$$

for any x and any $y \leq z$.

In view of Proposition 2.3.3, it is sufficient to show that

$$\min(\alpha+\beta-f(\alpha,\beta), \beta+\gamma-f(\beta,\gamma), \gamma+\alpha-f(\gamma,\alpha)) = \alpha+\beta-f(\alpha,\beta)\,,$$

for any $(\alpha,\beta,\gamma) \in \Delta$. As a double application of (2.27) leads to

$$\beta - f(\alpha,\beta) \leq \gamma - f(\alpha,\gamma)$$

and

$$\alpha - f(\beta,\alpha) \leq \gamma - f(\beta,\gamma)\,,$$

the proposition indeed holds. ∎

We now characterize the self-dual upper bound functions U_f of the form (2.26).

PROPOSITION – 2.3.5: *The minimum operator $T_{\mathbf{M}}$ is the only 1-Lipschitz commutative conjunctor $f \leq T_{\mathbf{M}}$ such that the associated upper bound function U_f is self-dual.*

Proof:
The self-dual upper bound functions of the form (2.26) are characterized by the equality

$$\alpha+\beta-f(\alpha,\beta)+1-\gamma+1-\beta-f(1-\gamma,1-\beta) = 1\,,$$

for any $(\alpha,\beta,\gamma) \in \Delta$. Rewriting this equality in the form

$$f(\alpha,\beta)+f(1-\gamma,1-\beta) = \alpha+(1-\gamma)\,,$$

and taking into account that $f \leq T_{\mathbf{M}}$, the only function f that identically satisfies this equality is $f = T_{\mathbf{M}}$. ∎

Note that the corresponding (self-dual) upper bound function is then simply given by $U_{\mathbf{M}}(\alpha,\beta,\gamma) = \alpha+\beta-\min(\alpha,\beta) = \beta$, as announced earlier. If we replace the condition $f \leq T_{\mathbf{M}}$ in Proposition 2.3.4 by the stronger condition (in the given context) that f should have 1 as neutral element, then we are in fact dealing with a commutative quasi-copula.

COROLLARY – 2.3.6: *Let C be a commutative quasi-copula. A probabilistic relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha,\beta,\gamma) = \alpha+\beta-C(\alpha,\beta)\,. \tag{2.28}$$

In case of a copula, the operation in (2.28) is known as the *dual of the copula* [64].

COROLLARY – 2.3.7: *Let C be a commutative copula. A probabilistic relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha,\beta,\gamma) = \tilde{C}(\alpha,\beta)\,, \tag{2.29}$$

where

$$\tilde{C}(\alpha,\beta) = \alpha+\beta-C(\alpha,\beta) \tag{2.30}$$

is the dual of the copula C.

Note that besides the dual of a copula, one also defines the *co-copula* C^* of a copula C by

$$C^*(x,y) = 1 - C(1-x, 1-y)\,, \tag{2.31}$$

and the *survival copula* $\hat{C}$ associated to the copula C by

$$\hat{C}(x,y) = x + y - 1 + C(1-x, 1-y)\,. \tag{2.32}$$

Neither the dual $\tilde{C}$, nor the co-copula C^* of a copula C is a copula [57]; on the other hand, the survival copula $\hat{C}$ associated to C is a copula.

Using this terminology, the dual lower bound function L_C can be written compactly as

$$\begin{aligned} L_C(\alpha,\beta,\gamma) &= 1 - U_C(1-\gamma, 1-\beta, 1-\alpha) \\ &= 1 - \tilde{C}(1-\gamma, 1-\beta) = \hat{C}(\gamma,\beta)\,. \end{aligned}$$

2.3.4 The case of t-norms

Corollary 2.3.7 applies in particular to t-norms that are copulas as well. Many parametric families of t-norms contain a subfamily of copulas [56]. On the other hand, there also exist lists of parametric families of copulas, most of them containing a parametric subfamily of t-norms [64].

Although they appear to be quite technical, the Frank t-norms (1.2) are important solutions of an often encountered functional equation. To that end, we first introduce the concept of an ordinal sum of t-norms [56].

PROPOSITION – 2.3.8: *Consider a countable family $(T_\alpha)_{\alpha\in A}$ of t-norms and a corresponding family $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$. The binary operation T on $[0,1]$ defined by*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) T_\alpha\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha}, \frac{y-a_\alpha}{e_\alpha-a_\alpha}\right) & \text{, if } (x,y) \in [a_\alpha, e_\alpha]^2\,, \\ \min(x,y) & \text{, otherwise.} \end{cases}$$

is a t-norm, and is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha\rangle$, $\alpha \in A$.

Ordinal sums of Frank t-norms were shown to be the only t-norms T solving the functional equation

$$T(x,y) + S(x,y) = x + y$$

for some t-conorm S. In particular, when $T = T^{\mathbf{F}}_\lambda$ this t-conorm is nothing else but the Frank t-conorm $S^{\mathbf{F}}_\lambda$ which coincides with the dual t-conorm of $T^{\mathbf{F}}_\lambda$ in the sense of (1.1):

$$S^{\mathbf{F}}_\lambda(x,y) = 1 - T^{\mathbf{F}}_\lambda(1-x, 1-y)\,. \tag{2.33}$$

In the latter case, Corollary 2.3.7 can be rephrased as follows.

PROPOSITION – 2.3.9: *A probabilistic relation Q on A is $T^{\mathbf{F}}_\lambda$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U^{\mathbf{F}}_\lambda$ defined by*

$$U^{\mathbf{F}}_\lambda(\alpha,\beta,\gamma) = S^{\mathbf{F}}_\lambda(\alpha,\beta)\,. \tag{2.34}$$

Note that due to (2.33), the dual lower bound function $L_\lambda^{\mathbf{F}}$ is given by

$$L_\lambda^{\mathbf{F}}(\alpha,\beta,\gamma) = T_\lambda^{\mathbf{F}}(\beta,\gamma)\,.$$

From Proposition 2.3.9 we obtain the following special cases.

(a) As was mentioned before, a probabilistic relation Q is $T_{\mathbf{M}}$-transitive if and only if it is cycle-transitive w.r.t. the self-dual upper bound function $U_{\mathbf{M}}$ defined by

$$U_{\mathbf{M}}(\alpha,\beta,\gamma) = \max(\alpha,\beta) = \beta\,. \tag{2.35}$$

Hence, for a $T_{\mathbf{M}}$-transitive probabilistic relation Q it must hold that $\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 = \beta_{abc}$, or equivalently, $\alpha_{abc} + \gamma_{abc} = 1$, for any $(a,b,c) \in A^3$.

(b) As proven in detail, a probabilistic relation Q is $T_{\mathbf{P}}$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U_{\mathbf{P}}$ defined by

$$U_{\mathbf{P}}(\alpha,\beta,\gamma) = \alpha + \beta - \alpha\beta\,. \tag{2.36}$$

(c) A probabilistic relation Q is $T_{\mathbf{L}}$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U_{\mathbf{L}}$ defined by

$$U_{\mathbf{L}}(\alpha,\beta,\gamma) = \min(\alpha+\beta,1)\,.$$

Hence, for a $T_{\mathbf{L}}$-transitive probabilistic relation Q it must hold that $\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq \min(\alpha_{abc} + \beta_{abc}, 1)$, for any $(a,b,c) \in A^3$. If $\alpha_{abc} + \beta_{abc} < 1$, then this inequality is trivially fulfilled. Therefore, a probabilistic relation Q is $T_{\mathbf{L}}$-transitive if and only if it is cycle-transitive w.r.t. the simpler equivalent upper bound function $U'_{\mathbf{L}}$ defined by

$$U'_{\mathbf{L}}(\alpha,\beta,\gamma) = 1\,. \tag{2.37}$$

Note that the same equivalence holds for the less elegant upper bound function $U''_{\mathbf{L}}$ defined by

$$U''_{\mathbf{L}}(\alpha,\beta,\gamma) = \begin{cases} 1 & ,\text{if } \beta \geq 1/2\,, \\ 2 & ,\text{if } \beta < 1/2\,. \end{cases} \tag{2.38}$$

Expressions (2.35)–(2.37) nicely illustrate that $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity and that $T_{\mathbf{P}}$-transitivity implies $T_{\mathbf{L}}$-transitivity.

2.4 Stochastic transitivity as cycle-transitivity

2.4.1 Stochastic transitivity

In this section, we propose a general notion of stochastic transitivity and show when and how it fits into the framework of cycle-transitivity.

DEFINITION – 2.4.1: *Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping. A probabilistic relation Q on A is called g-stochastic transitive if for any $(a,b,c) \in A^3$ it holds that*

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq g(Q(a,b), Q(b,c))\,. \tag{2.39}$$

This definition includes many well-known types of stochastic transitivity. Indeed, g-stochastic transitivity is known as

(i) strong stochastic transitivity when $g = \max$ [63];

(ii) moderate stochastic transitivity when $g = \min$ [63];

(iii) weak stochastic transitivity when $g = 1/2$ [63];

(iv) λ-transitivity, with $\lambda \in [0,1]$, when $g = \lambda \max + (1-\lambda)\min$ [4].

It is clear that strong stochastic transitivity implies λ-transitivity, which implies moderate stochastic transitivity, which, in turn, implies weak stochastic transitivity.

2.4.2 Stochastic transitivity as cycle-transitivity

PROPOSITION – 2.4.2: *Let g be a commutative, increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,x) \leq x$ for any $x \in [1/2,1]$. A probabilistic relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ \min(\alpha+\beta-g(\alpha,\beta), \beta+\gamma-g(\beta,\gamma), & \\ \qquad \gamma+\alpha-g(\gamma,\alpha)) & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.40}$$

Proof:
First of all, a tedious, yet simple verification shows that for a function g with the given properties the corresponding function U_g defined in (2.40) belongs to $\mathcal{U}$. Essential is the additional condition $g(1/2,x) \leq x$.

Consider a probabilistic relation Q on A and $(a,b,c) \in A^3$. If $\beta_{abc} \geq 1/2$, then also $\gamma_{abc} \geq 1/2$ and at least two of the three elements q_{ab}, q_{bc} and q_{ac} are greater than or equal to $1/2$. In this case, g-stochastic transitivity requires that $1-\alpha_{abc} \geq g(\beta_{abc},\gamma_{abc})$. If $\alpha_{abc} < 1/2$, this inequality is the only one that must hold for (a,b,c) (and cyclic permutations of it) and g-stochastic transitivity turns out to be equivalent to the condition:

$$\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq \beta_{abc}+\gamma_{abc}-g(\beta_{abc},\gamma_{abc})\,.$$

However, if $\alpha_{abc} \geq 1/2$, then two more inequalities must hold, namely $1-\gamma_{abc} \geq g(\alpha_{abc},\beta_{abc})$ and $1-\beta_{abc} \geq g(\alpha_{abc},\gamma_{abc})$, and these inequalities together yield the condition $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq \min(\alpha_{abc}+\beta_{abc}-g(\alpha_{abc},$

$\beta_{abc}), \beta_{abc}+\gamma_{abc}-g(\beta_{abc},\gamma_{abc}), \gamma_{abc}+\alpha_{abc}-g(\gamma_{abc},\alpha_{abc}))$. If $\beta_{abc} < 1/2$, there is no upper bound for $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1$, which means that we can just put 2. Summarizing, we have shown that g-stochastic transitivity can be reformulated as cycle-transitivity w.r.t. the upper bound function U_g. ∎

Note that in general the additional conditions (2.10) are not satisfied by an upper bound function of type (2.40). This is only the case when $g(1/2,1/2) > 0$ or $g(1/2,1) > 1/2$.

As in the case of fuzzy transitivity, we will establish sufficient conditions on the function g which allow to bring the upper bound function U_g in a simpler form. A first proposition restricts the range of g to the interval $[1/2,1]$. Cycle-transitivity w.r.t. U_g then always implies weak stochastic transitivity. Also, the additional conditions (2.10) are then trivially fulfilled.

PROPOSITION – 2.4.3: *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping such that $g(1/2,x) \leq x$ for any $x \in [1/2,1]$. A probabilistic relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.41}$$

Proof:
Consider a probabilistic relation Q on A and $(a,b,c) \in A^3$. In view of Proposition 2.4.2, we only need to consider the case $\alpha_{abc} \geq 1/2$ and we know that in this case g-stochastic transitivity requires that $1-\alpha_{abc} \geq g(\beta_{abc},\gamma_{abc})$, $1-\beta_{abc} \geq g(\alpha_{abc},\gamma_{abc})$ and $1-\gamma_{abc} \geq g(\alpha_{abc},\beta_{abc})$. Since g takes values in $[1/2,1]$, this can only hold if $\alpha_{abc} \leq 1/2$, $\beta_{abc} \leq 1/2$ and $\gamma_{abc} \leq 1/2$. Since $\alpha_{abc} \geq 1/2$ it then follows that $\alpha_{abc}=\beta_{abc}=\gamma_{abc}=1/2$. An equivalent way of arriving at this single possibility is by requiring that $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq 1/2$ in case $\alpha_{abc} \geq 1/2$. ∎

From Proposition 2.4.3 we obtain the following special cases.

(a) A probabilistic relation Q is strongly stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ss} defined by

$$U_{ss}(\alpha,\beta,\gamma) = \begin{cases} \beta & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.42}$$

(b) A probabilistic relation Q is moderately stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ms} defined by

$$U_{ms}(\alpha,\beta,\gamma) = \begin{cases} \gamma & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.43}$$

(c) A probabilistic relation Q is weakly stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ws} defined by

$$U_{ws}(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-1/2 & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.44}$$

(d) A probabilistic relation Q is λ-transitive, with $\lambda \in [0,1]$, if and only if it is cycle-transitive w.r.t. the upper bound function U_λ defined by

$$U_\lambda(\alpha,\beta,\gamma) = \begin{cases} \lambda\beta+(1-\lambda)\gamma & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.45}$$

A final simplification, eliminating the special case $\alpha = 1/2$ in (2.41), is obtained by requiring g to have neutral element $1/2$, i.e. $g(1/2,x) = g(x,1/2) = x$ for any $x \in [1/2,1]$.

PROPOSITION – 2.4.4: *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. A probabilistic relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.46}$$

Proof:
Consider a probabilistic relation Q on A and $(a,b,c) \in A^3$. As in the proof of Proposition 2.4.3, we only need to consider the case $\alpha_{abc} \geq 1/2$ in which g-stochastic transitivity is equivalent to $\alpha_{abc} = \beta_{abc} = \gamma_{abc} = 1/2$. We need to show that an equivalent way of arriving at this single possibility, knowing that g has neutral element $1/2$, is by requiring in this case that $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq \beta_{abc}+\gamma_{abc}-g(\beta_{abc},\gamma_{abc})$, or equivalently, $1-\alpha_{abc} \geq g(\beta_{abc},\gamma_{abc})$. Indeed, since g has neutral element $1/2$, it holds that $g \geq \max$, and we must have that $1-\alpha_{abc} \geq \gamma_{abc}$, which, given $\alpha_{abc} \geq 1/2$, only occurs when $\alpha_{abc} = \gamma_{abc} = 1/2$, whence also $\beta_{abc} = 1/2$. ∎

This proposition implies in particular that strong stochastic transitivity ($g = \max$) is equivalent to cycle-transitivity w.r.t. the simplified upper bound function U'_{ss} defined by

$$U'_{ss}(\alpha,\beta,\gamma) = \begin{cases} \beta & \text{, if } \beta \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{2.47}$$

Note that g-stochastic transitivity w.r.t. a function $g \geq \max$ always implies strong stochastic transitivity. This means that any probabilistic relation that is cycle-transitive w.r.t. an upper bound function U_g of the form (2.46) is at least strongly stochastic transitive.

Comparing (2.35) and (2.38) with (2.47) and (2.43), it is clear that $T_{\mathbf{M}}$-transitivity implies strong stochastic transitivity and that moderate stochastic transitivity implies $T_{\mathbf{L}}$-transitivity.

Note that there is no connection between the weakest form of stochastic transitivity (upper bound (2.44)) and the weakest form of C-transitivity (upper bound (2.38)).

2.4.3 Partial stochastic transitivity

In [38], the notion of partial stochastic transitivity is defined as

$$Q(a,b) > \frac{1}{2} \wedge Q(b,c) > \frac{1}{2} \Rightarrow Q(a,c) \geq \min(Q(a,b), Q(b,c)) . \qquad (2.48)$$

Here, we wish to extend this definition as follows.

DEFINITION – 2.4.5: *Let g be a commutative $]1/2,1]^2 \to]1/2,1]$ mapping. A probabilistic relation Q on A is called partially stochastic transitive w.r.t. the function g, or in short partially g-stochastic transitive, if for any $(a,b,c) \in A^3$ it holds that*

$$Q(a,b) > \frac{1}{2} \wedge Q(b,c) > \frac{1}{2} \Rightarrow Q(a,c) \geq g(Q(a,b), Q(b,c)) . \qquad (2.49)$$

Partial stochastic transitivity w.r.t. $g = \min$ will show up in Chapter 6 as the characteristic transitivity of uniformly distributed random variables coupled by the $T_{\mathbf{L}}$ copula. In the next proposition we present a way of representing partial g-stochastic transitivity as cycle-transitivity.

PROPOSITION – 2.4.6: *Let h be a commutative $[0,1]^2 \to [0,1]$ mapping such that $h(x,y) \leq 1/2$ if $\min(x,y) \leq 1/2$, and $h(x,y) > 1/2$ if $\min(x,y) > 1/2$. Let $g = h_{|]1/2,1]}$. It then holds that Q is partially g-stochastic transitive if and only if Q is cycle-transitive w.r.t. upper bound function*

$$U_h(\alpha,\beta,\gamma) = \beta + \gamma - h(\beta,\gamma) . \qquad (2.50)$$

Proof:

Consider a probabilistic relation Q on A and $(a,b,c) \in A^3$. We now investigate all possibilities, for ease of notation we drop the index abc in α_{abc}, β_{abc} and γ_{abc}.

Case 1: $\alpha > 1/2 \vee \gamma < 1/2$.

We consider the case $\alpha > 1/2$, the case $\gamma < 1/2$ is analogous.

(2.49) $\Rightarrow 1/2 > 1-\alpha \geq g(\beta,\gamma) > 1/2$, which is impossible. On the other hand (2.50) $\Rightarrow 1/2 < \alpha \leq 1 - h(\beta,\gamma) < 1/2$, which is also impossible.

Case 2: $\beta > 1/2 \wedge \alpha = 1/2 \vee \beta < 1/2 \wedge \gamma = 1/2$.

We consider the first possibility, the second possibility is analogous.

(2.49) $\Rightarrow 1/2 = 1-\alpha \geq g(\beta,\gamma) > 1/2$, again impossible. The same impossible condition is obtained using (2.50).

Case 3: $(\beta > 1/2 \wedge \alpha < 1/2) \vee (\beta < 1/2 \wedge \gamma > 1/2)$.
We again consider only the first possibility as the second possibility is analogous.

(2.50) $\Leftrightarrow \gamma \geq h(1-\beta, 1-\alpha) \wedge \alpha \leq 1 - h(\beta,\gamma) \Leftrightarrow \alpha \leq 1 - g(\beta,\gamma) \Leftrightarrow$ (2.49).
Case 4: $\beta = 1/2$.

Conditions (2.49) are always fulfilled. On the other hand, cycle-transitivity w.r.t. upper bound (2.50) is also always satisfied, thanks to the conditions upon h. ∎

Note that the conditions on the commutative function g in the above proposition are not only sufficient conditions, they are also necessary for the condition (2.49) to be equivalent to cycle-transitivity w.r.t. the upper bound function (2.50). The above proposition implies that partial g-stochastic transitivity can be equivalently expressed as cycle-transitivity w.r.t. upper bound function U_h such that $h(x,y) = g(x,y)$ when $\min(x,y) > 1/2$ and $h(x,y) = 1/2$ when $\min(x,y) \leq 1/2$. Note that the function $h = T_{\mathbf{P}}$ does not always satisfy the property $h(x,y) > 1/2$ when $\min(x,y) > 1/2$.

2.4.4 Isostochastic transitivity

In Subsection 2.2.1, we have derived the most general polynomial self-dual upper bound functions. Here we define another family of self-dual upper bound functions. Note that we will encounter multiple upper bound functions belonging to this new family.

PROPOSITION – 2.4.7: *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. It then holds that any $\Delta \to \mathbb{R}$ function U of the form*

$$U_g^s(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2\,, \\ \alpha+\beta-1+g(1-\beta,1-\alpha) & \text{, if } \beta < 1/2\,, \end{cases} \tag{2.51}$$

is a self-dual member of $\mathcal{U}$.

Proof:
When $\beta > 1/2$, it easily follows that the dual lower bound function $L(\alpha,\beta,\gamma)$ equals $\beta+\gamma-g(\beta,\gamma)$, and coincides with the upper bound function. When $\beta = 1/2$, both functions coincide provided that the equality

$$1/2+\gamma-g(1/2,\gamma) = \alpha-1/2+g(1/2,1-\alpha) \tag{2.52}$$

holds for any $\alpha \leq 1/2$ and $\gamma \geq 1/2$. This follows from the fact that $1/2$ is the neutral element of g. Finally, it should hold that $U(0,0,1) = 0$ and $U(0,1,1) = 1$. This is guaranteed by the fact that 1 is the absorbing element of g. Indeed, $g(x,1) \geq g(1/2,1) = 1$, and hence $g(1,1) = 1$. This concludes the proof that U is a self-dual member of $\mathcal{U}$. ∎

Note that condition (2.52) with $\alpha \leq 1/2 \leq \gamma$ is equivalent to the condition that $1/2$ is a neutral element of the function g defined over $[1/2,1]^2$. Indeed, rewriting (2.52) as $1-\alpha-g(1/2,1-\alpha) = g(1/2,\gamma)-\gamma$ we see that $1-\alpha-g(1/2,1-\alpha)$ must be fixed for any $\alpha \leq 1/2$. By setting $\alpha=\gamma=1/2$ in (2.52) we obtain that $g(1/2,1/2)=1/2$ and therefore $1-\alpha-g(1/2,1-\alpha)=0$ for $\alpha \leq 1/2$ which is equivalent to g having neutral element 1/2. The premisses in the above proposition are therefore well chosen.

Also note that the function g in Proposition 2.4.7 has the same properties as the function g in Proposition 2.4.4. Of course, the upper bound function U_g^s also satisfies the additional conditions (2.10).

Many of the polynomial self-dual upper bound functions can be recast in the form (2.51). For instance, the self-dual upper bound function $U_{\mathbf{M}}$ (which characterizes $T_{\mathbf{M}}$-transitivity) is of the form (2.51) with $g = \max$. As a second example, let us reconsider the case of the self-dual upper bound function $U_E(\alpha,\beta,\gamma) = \alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma$. Solving α (resp. γ) from the equation $\alpha+\beta+\gamma-1 = \alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma$ and substituting the solution in the expression for $U_E(\alpha,\beta,\gamma)$ in case $\beta \geq 1/2$ (resp. $\beta < 1/2$), we obtain the equivalent self-dual upper bound function

$$U'_E(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-\dfrac{\beta\gamma}{\beta\gamma+(1-\beta)(1-\gamma)} & \text{, if } \beta \geq 1/2\,, \\ \alpha+\beta-1+\dfrac{(1-\alpha)(1-\beta)}{\alpha\beta+(1-\alpha)(1-\beta)} & \text{, if } \beta < 1/2\,, \end{cases} \tag{2.53}$$

which is of the form (2.51) with g defined by

$$g(x,y) = \frac{x\,y}{x\,y+(1-x)(1-y)}\,. \tag{2.54}$$

It is also interesting to know how cycle-transitivity w.r.t. an upper bound function of type (2.51) relates to strong stochastic transitivity.

PROPOSITION – 2.4.8: *Cycle-transitivity w.r.t. an upper bound function of type* (2.51) *implies strong stochastic transitivity.*

Proof:
This follows immediately from the fact that $g(x,y) \geq \max(x,y)$. ∎

Surprisingly, cycle-transitivity w.r.t. an upper bound function of type (2.51) can be seen as a variant of g-stochastic transitivity, which is shown in the proposition below.

PROPOSITION – 2.4.9: *A probabilistic relation Q on A is cycle-transitive w.r.t. a self-dual upper bound function of type U_g^s if and only if for any $(a,b,c) \in A^3$ it holds that*

$$(Q(a,b) \geq 1/2 \;\wedge\; Q(b,c) \geq 1/2) \;\Rightarrow\; Q(a,c) = g(Q(a,b),Q(b,c))\,. \tag{2.55}$$

The probabilistic relation Q will also be called isostochastic transitive w.r.t. g, *or shortly,* g-isostochastic transitive.

Proof:
Consider any $(a,b,c) \in A^3$. When $\beta_{abc} > 1/2$, (2.51) is equivalent to $1-\alpha_{abc} = g(\beta_{abc}, \gamma_{abc})$. This is then equivalent to (2.55) as $\beta_{abc} \geq 1/2$ and $\gamma_{abc} \geq 1/2$ while $\beta_{cba} < 1/2$. When $\beta_{abc} < 1/2$, (2.51) is equivalent to $\gamma_{abc} = g(1-\alpha_{abc}, 1-\beta_{abc})$. This is again equivalent to (2.55) as then $1-\alpha_{abc} = \gamma_{cba} \geq 1/2$ and $1-\beta_{abc} = \beta_{cba} \geq 1/2$ while $\beta_{abc} < 1/2$. Finally, when $\beta_{abc} = 1/2$, conditions (2.51) and (2.55) are both equivalent to the condition $\alpha_{abc} = \gamma_{abc} = 1/2$. ∎

In particular, a probabilistic relation Q is $T_{\mathbf{M}}$-transitive if and only if

$$(Q(a,b) \geq 1/2 \ \wedge\ Q(b,c) \geq 1/2) \ \Rightarrow\ Q(a,c) = \max(Q(a,b), Q(b,c))\,,$$

for any $(a,b,c) \in A^3$. Note that this is formally the same as (1.9) with the difference that in the latter case Q was only $\{0, 1/2, 1\}$-valued.

Note that the properties we imposed on g in Propositions 2.4.4 and 2.4.7 are very close to the defining properties of t-conorms. Indeed, although associativity is not explicitly required, it follows quite naturally. Consider for instance a g-isostochastic transitive probabilistic relation Q such that $Q(a,b) \geq 1/2$, $Q(b,c) \geq 1/2$ and $Q(c,d) \geq 1/2$. Then it holds that

$$Q(a,d) = g(Q(a,b), Q(b,d)) = g(Q(a,b), g(Q(b,c), Q(c,d)))$$

and

$$Q(a,d) = g(Q(a,c), Q(c,d)) = g(g(Q(a,b), Q(b,c)), Q(c,d))\,,$$

whence at least for the triplet $(Q(a,b), Q(b,c), Q(c,d))$ the function g is associative. Adding (full) associativity makes g into a t-conorm on $[1/2, 1]$, or after appropriate rescaling, into a usual t-conorm on $[0,1]$.

PROPOSITION – 2.4.10: *When g is a commutative, associative and increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$, then the $[0,1]^2 \to [0,1]$ mapping S_g defined by*

$$S_g(x,y) = 2g\left(\frac{1+x}{2}, \frac{1+y}{2}\right) - 1$$

is a t-conorm.

Proof:
One easily verifies that since g is increasing, associative and commutative, also S_g is increasing, associative and commutative. Furthermore, S_g has 0 as neutral element since

$$S_g(0,x) = 2g(1/2, (1+x)/2) - 1 = (1+x) - 1 = x\,,$$

for any $x \in [0,1]$. ∎

The two examples of self-dual upper bound functions given above fall in the latter category. For the upper bound U_g^s with $g = \max$, corresponding to

$T_{\mathbf{M}}$-transitivity, we obtain $S_g = \max$. For the self-dual upper bound function U'_E in (2.53), the associated t-conorm S_E is given by

$$S_E(x, y) = \frac{x + y}{1 + xy}\,, \tag{2.56}$$

which is the Hamacher t-conorm $S^{\mathbf{H}}_2(x, y) = 1 - T^{\mathbf{H}}_2(1 - x, 1 - y)$ with parameter value 2 [48], where

$$S^{\mathbf{H}}_{\gamma}(x, y) = \frac{x + y - (2 - \gamma)xy}{1 - (1 - \gamma)xy}\,, \gamma \geq 0\,,$$

and

$$T^{\mathbf{H}}_{\gamma}(x, y) = \frac{xy}{\gamma + (1 - \gamma)(x + y - xy)}\,, \gamma \geq 0\,.$$

This *t*-conorm is a member of the well-known class of strict *t*-conorms which are of the form

$$S(x, y) = g^{-1}(g(x) + g(y))\,,$$

with g an additive generator, i.e. a strictly increasing and continuous $[0, 1] \to [0, +\infty]$ mapping that satisfies $g(0) = 0$ (see e.g. [57]). For $S^{\mathbf{H}}_2$, we obtain $g = \ln \frac{1+t}{1-t}$. For more properties about the Hamacher *t*-norms we refer to [39].

2.5 Symmetric payoff relations

The difference in complexity of (2.20) and (2.23) suggests that simpler transitivity formulations are obtained when the probabilistic relation $Q = [q_{ij}]$ is changed to the relation $Q' = [a_{ij}]$, $a_{ij} \in [-1/2, 1/2]$, using the transformation $a_{ij} = q_{ij} - 1/2$. The obtained values a_{ij} can be regarded as rescaled payoffs. In Chapters 5 and 7 a payoff matrix will be obtained by transforming the probabilistic relation in exactly this way. When the values a_{ij} describe a relation, we will call Q' a symmetric payoff relation.

Of course, the whole of probability theory could be restated by replacing the unit interval with the interval $[-1/2, 1/2]$. This will not be done here. We merely wish to restate cycle-transitivity when used for symmetric payoff relations and show how the upper bound functions are transformed. To that extent, let $\Delta' = \{(x, y, z) \in [-1/2, 1/2]^3 \mid x \leq y \leq z\}$ and let

$$\alpha'_{ijk} = \min(a_{ij}, a_{jk}, a_{ki})\,,\ \beta'_{ijk} = \mathrm{median}(a_{ij}, a_{jk}, a_{ki})\,,\ \gamma'_{ijk} = \max(a_{ij}, a_{jk}, a_{ki})\,.$$

The upper bound functions defined on Δ' are then given by those satisfying the definition below.

DEFINITION – 2.5.1: *A function $U' : \Delta' \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U'(-1/2, -1/2, 1/2) \geq -1/2$ and $U'(-1/2, 1/2, 1/2) \geq 1/2$;

(ii) for any $(\alpha', \beta', \gamma') \in \Delta'$:

$$U'(\alpha', \beta', \gamma') + U'(-\gamma', -\beta', -\alpha') \geq 0 \,. \tag{2.57}$$

The class of upper bound functions is denoted $\mathcal{U}'$.

The dual lower bound function L' of U' is given by

$$L'(\alpha', \beta', \gamma') = -U'(-\gamma', -\beta', -\alpha') \,. \tag{2.58}$$

Finally, cycle-transitivity for these symmetric payoff relations can be defined.

DEFINITION – 2.5.2: *A symmetric payoff relation Q on A is called cycle-transitive w.r.t. an upper bound function $U' \in \mathcal{U}'$ if for any $(a, b, c) \in A^3$ it holds that*

$$L'(\alpha'_{abc}, \beta'_{abc}, \gamma'_{abc}) \leq \alpha'_{abc} + \beta'_{abc} + \gamma'_{abc} \leq U'(\alpha'_{abc}, \beta'_{abc}, \gamma'_{abc}) \,, \tag{2.59}$$

where L' is the dual lower bound function of U'.

We end this section with a list of upper bound functions defined over Δ and their corresponding upper bound function defined over Δ'. To reduce the complexity of the table, we have removed the conditions on the values when an upper bound consists of multiple expressions. Note that the correspondence between both types of upper bound function is given by (2.18).

Table 2.1: *Correspondence between the upper bound functions of both definitions of cycle-transitivity.*

Name	Upper bound U	Upper bound U'
multiplicative	$\alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$	$-4\alpha'\beta'\gamma'$
polynomial	(2.23)	(2.20)
C-transitive	$\alpha + \beta - C(\alpha, \beta)$	$\alpha' + \beta' - C'(\alpha', \beta')$ $C'(x, y) = C(x + \frac{1}{2}, y + \frac{1}{2}) - \frac{1}{2}$
g-stochastic	$\begin{cases} \beta + \gamma - g(\beta, \gamma) \\ 1/2 \\ 2 \end{cases}$	$\begin{cases} \beta' + \gamma' - g'(\beta', \gamma') \\ 0 \\ 3/2 \end{cases}$ $g'(x, y) = g(x + \frac{1}{2}, y + \frac{1}{2}) - \frac{1}{2}$
g-isostochastic	$\begin{cases} \beta + \gamma - g(\beta, \gamma) \\ \alpha + \beta - 1 + \\ \quad g(1 - \beta, 1 - \alpha) \end{cases}$	$\begin{cases} \beta' + \gamma' - g'(\beta', \gamma') \\ \alpha' + \beta' - g'(-\beta', -\alpha') \end{cases}$ $g'(x, y) = g(x + \frac{1}{2}, y + \frac{1}{2}) - \frac{1}{2}$

A discovery is said to be an accident meeting a prepared mind.
— ALBERT VON SZENT-GYURGYI

3 The Dice Model

In this chapter, we introduce the notion of a dice model [32, 33] as a framework for generating a class of probabilistic relations. We investigate the transitivity of the probabilistic relations generated by a dice model and prove that it is a special type of cycle-transitivity, which we have called dice-transitivity, that is situated between moderate stochastic transitivity or product-transitivity on the one side, and Łukasiewicz-transitivity on the other side. Finally, it is shown that any probabilistic relation with rational elements on a 3-dimensional space of alternatives which possesses this particular type of cycle-transitivity, can be generated by a dice model. Dice-transitivity is a type of transitivity that cannot be cast into any of the classical types of transitivity, in which weak stochastic transitivity is a minimal requirement. As will be shown in this chapter, the probabilistic relation generated by a dice model are not all weakly stochastic transitive.

In Sections 1 and 2 the dice model is defined, Section 3 then involves a standardization of the specific dice used in the models, which will make the determination of the characteristic transitivity simpler. In Section 4, the characteristic transitivity of the dice model is determined. Section 5 then considers a natural question about the nature of dice-transitivity and shows that any dice-transitive 3-dimensional relation can be generated by a dice model consisting of at most seven so-called blocks. Finally, Section 6 investigates the transitivity of higher-dimensional dice models. It turns out that dice-transitivity is no longer the characteristic transitivity of the dice model for higher dimensions. Many efforts, of which we will report, have been made to obtain the desired result for 4-dimensional dice models, however no conclusive answer has yet been obtained. Note that the fact that dice-transitivity can be expressed in the framework of cycle-transitivity stresses the relevance of this newly developed tool for expressing transitivity properties.

3.1 Origin of the model

The dice model is inspired upon an ordinary game between two players, with three dice: Player 1 erases the spots from the faces of three fair dice (with 6 faces) and writes one number from 1, 2, ..., 18 on each face. Since Player 1 writes the numbers on the faces it seems fair to let Player 2 choose her dice first. Of course, Player 2 tries to choose the best dice. Each of them risks € 1, chooses one dice, they throw the dice, and the person having the bigger number on top of her dice receives the € 2. They can then throw their dice again until one of the players begins to show signs of boredom or is broke. It turns out that, despite the disadvantage of choosing last, Player 1 can distribute the numbers on the faces in such a way that she will always win in the long run.

Such an example for distributing the numbers over the three dice A, B, C is

$$A = \{1, 3, 4, 15, 16, 17\}, B = \{2, 10, 11, 12, 13, 14\}, C = \{5, 6, 7, 8, 9, 18\}.$$

Denoting by $P(X, Y)$ the probability that dice X wins from dice Y, we have $P(A, B) = 20/36$, $P(B, C) = 25/36$, $P(C, A) = 21/36$. In the above example, it holds that $P(A, B) > 1/2$, $P(B, C) > 1/2$ and $P(C, A) > 1/2$, which means that for any of the three dice, one of the remaining dice will always win from it in the long run. The corresponding relation "wins in the long run from" is therefore not transitive and forms a cycle. Formulating the above observation in another way, if we interprete the probabilities $P(A,B)$, $P(B,C)$ and $P(C,A)$ as elements of a $[0,1]$-valued relation on the space of alternatives $\{A,B,C\}$, then this valued relation is even not weakly stochastic transitive.

The above example will be generalized to the concept of a dice model in the following sense. Firstly, it is possible to consider an arbitrary (but fixed) number $m \geq 2$ of dice, each dice being characterized by a set A_i $(i = 1, 2, \ldots, m)$ of integers. The value m then denotes the dimension of the dice model. Secondly, each set A_i may contain n_i integers, with n_i not necessarily equal to six. In other words, we allow a dice to possess any number of faces, but do not care whether such a dice can be materialized and we will nevertheless maintain the metaphor dice. Finally, we do not insist on having mutually distinct numbers on the faces of a single dice or among different dice. We do impose the numbers to be strictly positive integers as this seems appropriate in the context of dice. Allowing negative numbers would not add new probabilistic relations as there always exists an equivalent model, w.r.t. the transitivity of the probabilistic relation, in which only positive integers are used. Indeed, it suffices to increment each element from the original model containing negative numbers by the absolute value of the lowest number with an additional increment of 1. Similarly, allowing real numbers does not add new probabilistic relations either. In the present chapter we are only concerned about the transitivity properties of the models and therefore only the relative order of the elements matters. However, in chapters 5 and 7 the restriction of allowing only strictly positive integers will be important, as the sum of the integers on the faces of a dice will become important.

Given a set of m generalized dice we will define the winning probabilities for each pair of dice and this set of dice is called an m-dimensional dice model, generating the corresponding probabilistic relation.

3.2 The dice model

In the dice model, the name dice is reserved to denote a finite multiset of strictly positive integers. We recall that a multiset is a set in which the elements need not be distinct. The cardinality of the multiset equals the number of faces of the dice and on each face is written exactly one of the numbers from the multiset so that each number appears on exactly one face. Furthermore, each face of the dice has equal likelihood of showing up when the corresponding hypothetical material dice is randomly thrown. Throughout this work, the terms dice and multiset will be used for denoting the same concept, explained above.

When a total of m dice will be compared, we speak of a *collection* of m dice. When it is clear that a collection of dice or multisets is meant, we will simply use the term collection. The *collective multiset* corresponding to a collection of m dice is given by the union of the multisets corresponding to the dice. The cardinality of this collective multiset therefore equals the sum of the cardinalities of those m multisets. For the purpose of this chapter, we will frequently make use of a special type of collection, called standard collection. This type of collection will also be of interest in Chapter 8.

DEFINITION – 3.2.1: *A standard collection is a collection of m multisets M_i of cardinality n_i for which the collective multiset M is given by $\mathbb{N}[1, n_1 + n_2 + \cdots + n_m]$.*

Throughout this work, $\mathbb{N}[a, b]$ will denote the set of integers in the interval $[a, b]$. When $a > b$ then $[a, b] = \emptyset$. Note that we will always write the elements of a multiset nondecreasingly. Definition 3.2.1 implies that all elements of the collective multiset M are different and that every integer in M occurs once in just one of the composing multisets M_i. In fact, the multisets M_i of a standard collection are ordinary sets that constitute a partition of the ordinary set M.

We now indicate how we can unambiguously associate a probabilistic relation to a given collection of dice.

DEFINITION – 3.2.2: *For any two dice A and B, with n_1 resp. n_2 faces, we define*

$$P(A,B) = \frac{1}{n_1 n_2} \left(\#\{(a,b) \in A \times B \mid a > b\}\right) ,$$

and

$$I(A,B) = \frac{1}{n_1 n_2} \left(\#\{(a,b) \in A \times B \mid a = b\}\right) .$$

It then holds that

$$D(A,B) = P(A,B) + \frac{1}{2}\, I(A,B) \tag{3.1}$$

is a probabilistic relation.

It should be noted that, given a couple (A, B) of multisets, $P(A, B)$ (resp. $I(A, B)$) is the probability that an element drawn at random (with a uniform distribution) from the multiset A is strictly greater than (resp. equal to) an element drawn at random from the multiset B. If, for example, A is an ordinary integer set of cardinality n, then according to the above definition, we obtain $P(A, A) = (n-1)/2n$ and $I(A, A) = 1/n$. In the context of fuzzy preference modelling [20], a strict preference relation P is assumed to be irreflexive ($P(A, A) = 0$) and an indifference relation to be reflexive ($I(A, A) = 1$). The $[0, 1]$-valued relations introduced above, despite their probabilistic interpretation, do not fit into the framework of fuzzy preference structures. However, the probabilistic relation D can also be written as

$$D(A,B) = P'(A,B) + \frac{1}{2}\, I'(A,B) ,$$

where P' and I' are defined by

$$\begin{aligned} P'(A,B) &= \max(P(A,B) - P(B,A), 0) , \\ I'(A,B) &= 1 - \mid P(A,B) - P(B,A) \mid . \end{aligned}$$

Now, P' (resp. I') can be interpreted as a strict preference (resp. indifference) relation. In particular, for an ordinary integer set A of any cardinality n, we obtain $P'(A,A) = 0$ and $I'(A,A) = 1$.

DEFINITION – 3.2.3: *The probabilistic relation $Q = [q_{ij}]$ generated by an m-dimensional dice model consisting of the collection of dice $(A_1, A_2, \ldots, A_m)$ is defined by $q_{ij} = Q(A_i, A_j) = D(A_i, A_j)$, with D defined in* (3.1).

The probabilistic relation generated by an m-dimensional dice model can be represented by a weighted directed graph with m nodes. Node i corresponds to dice A_i. Between every pair of nodes an arc is drawn and its direction is arbitrarily chosen. If an arc is drawn from node i to node j, then it carries the weight q_{ij}. It may be replaced by an arc from node j to node i carrying the weight $q_{ji} = 1 - q_{ij}$. Since $q_{ii} = 1/2$ for all i, for the sake of simplicity, loops at the graph nodes are not drawn. Figure 3.1 illustrates the graphical representation of the probabilistic relation generated by a dice model.

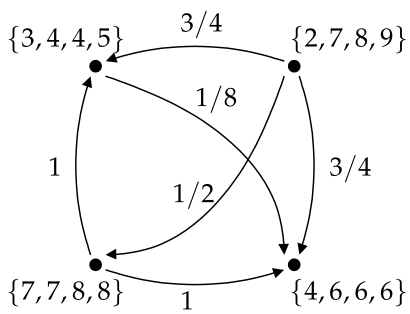


Figure 3.1: *A 4-node graph representing the probabilistic relation generated by the dice* $\{3,4,4,5\}$, $\{2,7,8,9\}$, $\{4,6,6,6\}$ *and* $\{7,7,8,8\}$.

3.3 Standardization of a dice model

From the definitions introduced in the previous section it is clear that many different collections of m multisets can generate the same probabilistic relation. The question arises whether for a probabilistic relation generated by a collection of multisets, there always exists at least one standard collection that generates the same probabilistic relation. An affirmative answer to this question is obtained in this section.

LEMMA – 3.3.1: *Any collection $C = (A_1, A_2, \ldots, A_m)$, with collective multiset M, can be transformed into a collection $C' = (A'_1, A'_2, \ldots, A'_m)$, where $\#A_i = \#A'_i$, with collective multiset M', so that:*

$$\begin{array}{l} 1.\ C \text{ and } C' \text{ generate the same probabilistic relation;} \\ 2.\ 1 \in M'; \\ 3.\ v \in M' \Rightarrow v \in \mathbb{N}[1, n_1 + n_2 + \ldots + n_m]; \\ 4.\ v \text{ occurs } n \text{ times in } M' \Rightarrow M' \cap \mathbb{N}[v+1, v+n-1] = \emptyset. \end{array} \tag{3.2}$$

Proof:
The proposed transformation of C into C' is essentially the unique order-preserving renumbering of the elements of M satisfying conditions 2, 3 and 4. ■

EXAMPLE – 3.3.2:
To illustrate this transformation, let us consider the following example. The collection (A_1, A_2, A_3) with

$$\begin{cases} A_1 = \{2, 2, 11, 14, 15\} \\ A_2 = \{2, 3, 3, 5, 12\} \\ A_3 = \{8, 8, 8, 9, 10\} \end{cases}$$

is transformed into the collection (A'_1, A'_2, A'_3) with

$$\begin{cases} A'_1 = \{1, 1, 12, 14, 15\} \\ A'_2 = \{1, 4, 4, 6, 13\} \\ A'_3 = \{7, 7, 7, 10, 11\} \end{cases}$$

One can easily verify that $Q(A_1, A_2) = Q(A'_1, A'_2) = 3/5$, that $Q(A_2, A_3) = Q(A'_2, A'_3) = 1/5$ and that $Q(A_3, A_1) = Q(A'_3, A'_1) = 2/5$. ◀

THEOREM – 3.3.3: *Any collection $C = (A_1, A_2, \ldots, A_m)$ can be transformed into a standard collection $\tilde{C} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m)$, where $\#\tilde{A}_i = 2\#A_i$, that generates the same probabilistic relation.*

Proof:
We first transform C into C', using Lemma 3.3.1. Next, we will transform the collective multiset M' of the collection C' into a multiset $\tilde{M}$ corresponding to a standard collection $\tilde{C} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m)$, with $\#\tilde{A}_i = 2\#A'_i$, that generates the same probabilistic relation.

For each distinct number ℓ in the multisets of C' (each distinct number in M') we do the following. If ℓ occurs only once in M', then we replace it by $2\ell - 1$ and 2ℓ. So, the multiset $\tilde{A}_i$ of $\tilde{C}$ that corresponds to the multiset A'_i of C' containing ℓ, contains $2\ell - 1$ and 2ℓ instead. If the number ℓ occurs twice in M', we replace one ℓ by $2\ell - 1$ and $2\ell + 2$ and the other ℓ by 2ℓ and $2\ell + 1$. Generally speaking, if ℓ occurs t times in M', then we replace the j^{th} ℓ, $j \geq 1$, by $2\ell + j - 2$ and $2\ell + 2t - j - 1$. Note that the t equal numbers ℓ are arbitrarily ordered, each ordering possibly giving rise to a different standard collection.

We will now prove that C' and $\tilde{C}$ generate the same probabilistic relation. As a first step we note that, thanks to the fourth property in (3.2), for any two distinct numbers $a > b$ from M' that are respectively transformed into the pairs of numbers a_1, a_2 and b_1, b_2 contained in $\tilde{M}$, it holds that both a_1 and a_2 are strictly greater than b_1 and b_2. Therefore the contribution to $D(A'_i, A'_j)$ originating from different numbers in A'_i and A'_j (the $P(A'_i, A'_j)$ part) equals the contribution to $D(\tilde{A}_i, \tilde{A}_j)$ originating from the transformed pairs of those numbers in $\tilde{A}_i$ and $\tilde{A}_j$. It remains to investigate whether the $I(A'_i, A'_j)$ contribution to $D(A'_i, A'_j)$ is reproduced by the transformed numbers (which are mutually distinct). To that aim let us consider the case where ℓ occurs in at most 2 multisets, say A'_i and A'_j, k times in A'_i and $t-k$ times in A'_j with $k \in \{0, 1, \ldots, t\}$ and $t > 1$. Without loss of generality we can assume that no other number but ℓ occurs in A'_i and A'_j and that $\ell = 1$.

According to the proposed transformation, $\tilde{A}_i$ contains the $2k$ numbers j_1, $2t - j_1 + 1$, j_2, $2t - j_2 + 1$, $\ldots$, j_k, $2t - j_k + 1$, with $1 \leq j_1 < j_2 < \cdots < j_k \leq t$, whereas $\tilde{A}_j$ contains the remaining numbers in $\mathbb{N}[1, 2t]$. Counting the number s of couples $(a, b) \in \tilde{A}_i \times \tilde{A}_j$ for which $a > b$, we obtain in increasing order of a:

$$
\begin{aligned}
s &= (j_1 - 1) + (j_2 - 2) + \cdots + (j_k - k) + \\
&\quad (2t - j_k - k) + (2t - j_{k-1} - k - 1) + \cdots + (2t - j_1 - 2k + 1) \\
&= 2k(t - k)\,.
\end{aligned}
$$

Hence, $D(\tilde{A}_i, \tilde{A}_j) = 2k(t-k)/(4k(t-k)) = 1/2$ which is equal to $D(A'_i, A'_j)$. Finally, the generalization to the case where the same number ℓ occurs in three or more multisets is straightforward. ∎

EXAMPLE – 3.3.4:
Continuing the same example as before, the standard collection (A'_1, A'_2, A'_3) can for instance be transformed into the collection $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ with

$$
\begin{cases}
\tilde{A}_1 = \{2, 3, 4, 5, 23, 24, 27, 28, 29, 30\} \\
\tilde{A}_2 = \{1, 6, 7, 8, 9, 10, 11, 12, 25, 26\} \\
\tilde{A}_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22\}
\end{cases}
$$

One can easily verify that the generated probabilistic relation is unchanged. ◄

Theorem 3.3.3 enables us to focus without loss of generality solely upon standard collections when investigating the transitivity properties of the probabilistic relations generated by collections of multisets. We will call standard collections of two (resp. three, resp. four) dice standard duplets (resp. standard triplets, resp. standard quartets). Note that for standard collections it holds that $D(A_i, A_j) = P(A_i, A_j)$ for all $i \neq j$.

3.4 Transitivity of the dice model

In this section, we will use the concept of cycle-transitivity to show that the type of transitivity exhibited by a probabilistic relation Q generated by a dice model, can be situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity. This is expressed in the next four theorems.

PROPOSITION – 3.4.1: *Not all probabilistic relations that are generated by a dice model are $T_{\mathbf{P}}$-transitive.*

Proof:
The probabilistic relation generated by the dice from Example (3.3.4) is not $T_{\mathbf{P}}$-transitive as it holds that $Q(A_2, A_1) = 2/5$, $Q(A_1, A_3) = 3/5$, $Q(A_2, A_3) = 1/5$ and $2/5 \cdot 3/5 > 1/5$. ■

Since $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity, clearly not all probabilistic relations generated by a dice model are $T_{\mathbf{M}}$-transitive.

PROPOSITION – 3.4.2: *Every probabilistic relation generated by a dice model is $T_{\mathbf{L}}$-transitive.*

Proof:
Firstly, we note that in view of Theorem 3.3.3 the proof must only be given for an arbitrary standard collection. Furthermore, we only need to show that the elements of the generated probabilistic relation $Q = [q_{ij}]$ satisfy the double inequality

$$0 \leq \alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 \leq 1$$

for all $i < j < k$. Let us define

$$x_{ijk} = \frac{1}{n_i n_j n_k} \#\{(x_i, x_j, x_k) \in A_i \times A_j \times A_k \mid x_i > x_j > x_k\} ,$$

then, since the collection is standard, it follows that

$$x_{ijk} + x_{ikj} + x_{jik} + x_{jki} + x_{kij} + x_{kji} = 1 .$$

On the other hand, it holds that $q_{ij} = x_{ijk} + x_{ikj} + x_{kij}$, $q_{jk} = x_{ijk} + x_{jik} + x_{jki}$, and $q_{ki} = x_{kij} + x_{kji} + x_{jki}$. Consequently,

$$\alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 = q_{ij} + q_{jk} + q_{ki} - 1 = x_{ijk} + x_{jki} + x_{kij} ,$$

and the value of the last expression always lies in $[0, 1]$, which completes the proof. ■

The reverse statement is not always true, as is illustrated by the following proposition.

PROPOSITION – 3.4.3: *Not all $T_{\mathbf{L}}$-transitive probabilistic relations can be generated by a dice model.*

Proof:
We will indicate a family of 3-dimensional $T_{\mathbf{L}}$-transitive probabilistic relations that cannot be generated by a triplet of dice. Indeed, let us consider the 3-dimensional probabilistic relation Q with rational elements satisfying $q_{12} \neq 1$, $q_{23} \neq 1$, $q_{31} \neq 1$ and $q_{12} + q_{23} + q_{31} = 2$. Clearly such relations exist and are $T_{\mathbf{L}}$-transitive. Suppose that there exists a standard triplet (A_1, A_2, A_3) with $\#A_1 = n_1$, $\#A_2 = n_2$ and $\#A_3 = n_3$, such that

$$q_{12} + q_{23} + q_{31} = 2 .$$

Let us first consider the case where the largest number $n = n_1 + n_2 + n_3$ is not in a multiset of cardinality one. Without loss of generality we can assume that A_1 contains the number n. Let us consider the standard triplet (A'_1, A'_2, A'_3), with corresponding elements q'_{12}, q'_{23} and q'_{31} of the probabilistic relation Q', that is obtained from (A_1, A_2, A_3) after removing n from A_1. Hence $\#A'_1 = n_1 - 1$ and we have in particular:

$$\begin{aligned} q'_{12} &= \frac{n_1 n_2 q_{12} - n_2}{(n_1-1)n_2} = \frac{n_1 q_{12} - 1}{n_1 - 1} = q_{12} + \frac{q_{12}-1}{n_1-1}\,, \\ q'_{23} &= q_{23}\,, \\ q'_{31} &= \frac{n_3 n_1 q_{31}}{n_3(n_1-1)} = \frac{n_1 q_{31}}{n_1-1} = q_{31} + \frac{q_{31}}{n_1-1}\,. \end{aligned}$$

It follows that

$$\begin{aligned} q'_{12} + q'_{23} + q'_{31} &= q_{12} + q_{23} + q_{31} + \frac{q_{12} + q_{31} - 1}{n_1 - 1} \\ &= 2 + \frac{1 - q_{23}}{n_1 - 1} > 2\,, \end{aligned}$$

which is a contradiction since by Proposition 3.4.2 the above sum should not exceed 2 for a standard triplet. Therefore, (A_1, A_2, A_3) is not a standard triplet.

There remains the case of a standard triplet with n contained in a multiset of cardinality 1. Suppose $n_1 = 1$ and $A_1 = \{n\}$ with $n = 1 + n_2 + n_3$. It follows that $q_{31} = 0$ and $q_{12} = 1$, but the latter equality is clearly not in agreement with the basic assumptions. Finally, since Q cannot be generated by a standard triplet, due to Theorem 3.3.3, it cannot be generated by an arbitrary triplet. ■

In the case of $T_{\mathbf{P}}$-transitive probabilistic relations, we can give conditions under which their generation by means of a dice model is always possible.

PROPOSITION – 3.4.4: *Every 3-dimensional $T_{\mathbf{P}}$-transitive probabilistic relation Q with rational elements can be generated by a dice model.*

Proof:
Consider a 3-dimensional probabilistic relation Q with rational elements q_{ij}. Without loss of generality (the other case is analogous), we assume that the elements of Q can be relabelled such that

$$q_{12} = \alpha_{123},\ q_{23} = \beta_{123},\ q_{31} = \gamma_{123}. \tag{3.3}$$

Since α_{123}, β_{123}, γ_{123} are rational numbers, they have a least common denominator which we will call n. Furthermore, let $p = n\,\alpha_{123}$, $q = n\,\beta_{123}$ and $r = n\,\gamma_{123}$. In this notation, $T_{\mathbf{P}}$-transitivity means that the double inequality

$$qr \le n(p+q+r-n) \le n(p+q) - pq$$

holds. Since $qr \le nq \le n(p+q) - pq$, we can distinguish two cases for the construction of the standard triplet. The first case is the one where p, q, r satisfy

$$qr \le n(p+q+r-n) \le nq\,,$$

or, equivalently,

$$(n-q)(n-r) \leq np \leq n(n-r)\,. \tag{3.4}$$

Then we define

$$\begin{aligned} A_1 &= \mathbb{N}[1,r] \cup E\,, \\ A_2 &= \mathbb{N}[r+1, n-q+r] \cup E^c\,, \\ A_3 &= \mathbb{N}[n-q+r+1, 2n-q+r]\,, \end{aligned} \tag{3.5}$$

with E a subset with cardinality $(n-r)$ of $\mathbb{N}[2n-q+r+1,3n]$ and $E^c = \mathbb{N}[2n-q+r+1,3n] \setminus E$. E^c has cardinality q. From (3.5) it is immediately clear that $Q(A_2,A_3) = nq/n^2 = \beta_{123}$ and $Q(A_3,A_1) = nr/n^2 = \gamma_{123}$. Depending upon the choice of E we obtain that $Q(A_1,A_2)$ can vary in steps of $1/n^2$ from $(n-q)(n-r)/n^2$ when $E = \mathbb{N}[2n-q+r+1,3n-q]$ to $n(n-r)/n^2$ when $E = \mathbb{N}[2n+r+1,3n]$. In particular, for all p satisfying (3.4) at least one subset E can be found for which $Q(A_1,A_2) = np/n^2 = \alpha_{123}$.

The second case is the one where p, q, r satisfy

$$nq \leq n(p+q+r-n) \leq n(p+q) - pq\,,$$

or, equivalently,

$$n(n-p) \leq nr \leq n^2 - pq\,. \tag{3.6}$$

We now define

$$\begin{aligned} A_1 &= \mathbb{N}[1,n-p] \cup E^c\,, \\ A_2 &= \mathbb{N}[n-p+q+1, 2n-p+q]\,, \\ A_3 &= \mathbb{N}[n-p+1, n-p+q] \cup E\,, \end{aligned} \tag{3.7}$$

where E is a subset with cardinality $(n-q)$ of $\mathbb{N}[2n-p+q+1,3n]$ and therefore $E^c = \mathbb{N}[2n-p+q+1,3n] \setminus E$ has cardinality p. From (3.7) it is immediately clear that $Q(A_1,A_2) = np/n^2 = \alpha_{123}$ and $Q(A_2,A_3) = nq/n^2 = \beta_{123}$. Depending upon the choice of E we obtain that $Q(A_3,A_1)$ can vary from $n(n-p)/n^2$ when $E = \mathbb{N}[2n-p+q+1,3n-p]$ to $(n(n-p)+(n-q)p)/n^2 = (n^2-pq)/n^2$ when $E = \mathbb{N}[2n+q+1,3n]$. Hence for all r satisfying (3.6) again at least one subset E can be found for which $Q(A_3,A_1) = nr/n^2 = \gamma_{123}$. ∎

Since $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity, the construction from Proposition 3.4.4 can be used to establish a standard triplet that generates a given 3-dimensional $T_{\mathbf{M}}$-transitive probabilistic relation with rational elements. Using the same notations as before and using the property $p + r = n$ which characterizes $T_{\mathbf{M}}$-transitivity, we obtain two constructions, namely

$$\begin{aligned} A_1 &= \mathbb{N}[1,r] \cup \mathbb{N}[2n+r+1,3n]\,, \\ A_2 &= \mathbb{N}[r+1,n-q+r] \cup \mathbb{N}[2n-q+r+1,2n+r]\,, \\ A_3 &= \mathbb{N}[n-q+r+1,2n-q+r]\,, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} A_1 &= \mathbb{N}[1,r] \cup \mathbb{N}[2n+r+1,3n]\,, \\ A_2 &= \mathbb{N}[q+r+1,n+q+r]\,, \\ A_3 &= \mathbb{N}[r+1,q+r] \cup \mathbb{N}[n+q+r+1,2n+r]\,. \end{aligned} \tag{3.9}$$

Using the equality $\alpha_{123}+\gamma_{123}=1$, we indeed obtain for both standard triplets that $q_{12}=Q(A_1,A_2)=\alpha_{123}$, $q_{23}=Q(A_2,A_3)=\beta_{123}$ and $q_{31}=Q(A_3,A_1)=\gamma_{123}$.

We now want to characterize more precisely the transitivity of the probabilistic relations generated by a dice model and therefore have reached the point where the most important results of this chapter can be formulated.

THEOREM – 3.4.5: *Every probabilistic relation generated by a dice model is cycle-transitive w.r.t. the upper bound function U_D defined by*

$$U_D(\alpha,\beta,\gamma)=\beta+\gamma-\beta\gamma\,. \tag{3.10}$$

Cycle-transitivity w.r.t. the upper bound function U_D will be called dice-transitivity.

Proof:
We are not able to formulate a direct proof in the style of the one of Proposition 3.4.2. Instead, we will establish a proof by induction. As said before, we can restrict the proof to probabilistic relations Q generated by a standard triplet.

By induction, suppose there exists a standard triplet (A_1,A_2,A_3) for which $\#A_1=n_1$, $\#A_2=n_2$, $\#A_3=n_3$ and that generates a dice-transitive probabilistic relation, i.e.

$$\alpha\beta \leq \alpha+\beta+\gamma-1\,, \tag{3.11}$$

for both loop directions.

We now construct a new standard triplet by attributing to one of the three multisets the additional number $n=n_1+n_2+n_3+1$. We can arbitrarily attribute n to A_1, because there have not been put any restrictions on the three multisets. We need to prove that for this new triplet the inequality (3.11) still holds for both loop directions. We will give the proof for the 123-loop. The proof for the 321-loop is completely similar and is left to the reader. For the values α_{123}, β_{123}, γ_{123} we will drop the indices. We will denote the newly obtained values, after attributing an additional number, with accents and we obtain:

$$\begin{aligned} n_1' &= n_1+1\,, \quad n_2'=n_2\,, \quad n_3'=n_3\,, \\ q_{12}' &= \frac{n_1n_2q_{12}+n_2}{(n_1+1)n_2}=\frac{n_1q_{12}+1}{n_1+1}\,, \\ q_{23}' &= q_{23}\,, \\ q_{31}' &= \frac{n_1q_{31}}{n_1+1}\,, \end{aligned}$$

from which it follows that:

$$q'_{12} + q'_{23} + q'_{31} - 1 = \frac{n_1}{n_1+1}(\alpha+\beta+\gamma-1) + \frac{q_{23}}{n_1+1}\,. \tag{3.12}$$

We distinguish three cases for q_{23} and in each case the induction hypothesis (3.11) is utilized.

Case 1: $q_{23} = \alpha$.

$(3.12) \geq \alpha\frac{n_1\beta+1}{n_1+1} = q'_{23}\frac{n_1\beta+1}{n_1+1} \geq \alpha'\beta'\,.$

Case 2: $q_{23} = \beta$.

$(3.12) \geq \beta\frac{n_1\alpha+1}{n_1+1} = q'_{23}\frac{n_1\alpha+1}{n_1+1} \geq \alpha'\beta'\,.$

Case 3: $q_{23} = \gamma$.

Case 3.1: $q_{31} = \alpha$.

$(3.12) \geq \alpha\frac{n_1\beta+1}{n_1+1} = \alpha q'_{12} \geq q'_{31}q'_{12} \geq \alpha'\beta'\,.$

Case 3.2: $q_{31} = \beta$.

$(3.12) \geq \beta\frac{n_1\alpha+1}{n_1+1} = \beta q'_{12} \geq q'_{31}q'_{12} \geq \alpha'\beta'\,.$

Finally, we still need to start the induction and therefore have to consider the basic case, which according to the induction hypothesis consists of a standard triplet where the multiset containing the highest number n is a singleton. We need to prove that for such a triplet, inequality (3.11) holds for both loop directions. For both loop directions it clearly holds that $\alpha = 0$ and $\gamma = 1$. Therefore, it is sufficient that for both loop directions $0 \leq 0 + \beta + 1 - 1$ holds and this inequality is indeed always satisfied. ∎

Dice-transitivity is a weaker type of transitivity than $T_{\mathbf{P}}$-transitivity, but is stronger than $T_{\mathbf{L}}$-transitivity. This follows from the fact that

$$U_{\mathbf{P}}(\alpha,\beta,\gamma) = \beta + \alpha(1-\beta) \leq \beta + \gamma(1-\beta) = U_D(\alpha,\beta,\gamma)\,,$$

and

$$U_D(\alpha,\beta,\gamma) = 1-(1-\beta)(1-\gamma) \leq 1 = U'_{\mathbf{L}}(\alpha,\beta,\gamma)\,.$$

Although $U_{ms} \not\leq U_D$, it does hold that moderate stochastic transitivity implies dice-transitivity. We first prove a more general proposition.

PROPOSITION – 3.4.6: *Cycle-transitivity w.r.t. an upper bound function defined by* $U(\alpha,\beta,\gamma) = \beta+\gamma-g(\beta,\gamma)$*, with* g *a* $[1/2,1]^2 \to [0,1]$ *mapping and* $g \leq \min$*, is equivalent to cycle-transitivity w.r.t. upper bound*

$$U'(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta > 1/2, \\ 2 & \text{, otherwise.} \end{cases} \tag{3.13}$$

Proof:
It is easily verified that for $\beta \leq 1/2$, the upper bound condition of cycle-transitivity w.r.t. U is always fulfilled. ∎

As $xy \leq \min(x, y)$, dice-transitivity is equivalent to cycle-transitivity w.r.t.

$$U'_D(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - \beta\gamma & \text{, if } \beta > 1/2, \\ 2 & \text{, otherwise.} \end{cases}$$

It is now easily verified that $U_{ms} \leq U'_D$ and therefore moderate stochastic transitivity implies dice-transitivity.

It is an interesting result that under the same conditions as for $T_{\mathbf{P}}$-transitive probabilistic relations, also dice-transitive probabilistic relations can be generated by a dice model. In the next chapter, it will even be shown that this extra condition of the rationality of the elements of the probabilistic relations is superfluous.

THEOREM – 3.4.7: *Every 3-dimensional dice-transitive probabilistic relation Q with rational elements can be generated by a dice model.*

Proof:
The proof closely resembles the proof of Proposition 3.4.4. We again consider the case where

$$q_{12} = \alpha_{123},\ q_{23} = \beta_{123},\ q_{31} = \gamma_{123},$$

and let $n,\ p,\ q,\ r$ denote the same quantities as before. Hence, with these notations dice-transitivity means that the double inequality

$$pq \leq n(p + q + r - n) \leq n(q + r) - qr$$

holds. As $pq \leq nq \leq n(q + r) - qr$, we will again distinguish two cases for the construction of the standard triplet (A_1, A_2, A_3). The first case is the one where $p,\ q,\ r$ satisfy

$$pq \leq n(p + q + r - n) \leq nq\,,$$

or, equivalently,

$$(n - p)(n - q) \leq nr \leq n(n - p)\,. \tag{3.14}$$

Then we define

$$\begin{aligned} A_1 &= E^c \cup \mathbb{N}[3n - p + 1, 3n]\,, \\ A_2 &= \mathbb{N}[n - p + q + 1, 2n - p + q]\,, \\ A_3 &= E \cup \mathbb{N}[2n - p + q + 1, 3n - p]\,, \end{aligned} \tag{3.15}$$

with E a subset with cardinality q of $\mathbb{N}[1, n - p + q]$ and $E^c = \mathbb{N}[1, n - p + q] \setminus E$. Note that E^c has cardinality $(n - p)$. From (3.15) it is immediately clear that $Q(A_1, A_2) = np/n^2 = \alpha_{123}$ and $Q(A_2, A_3) = nq/n^2 = \beta_{123}$. Depending upon the choice of E we obtain that $Q(A_3, A_1)$ can vary in steps of $1/n^2$ from $(n - p)(n - q)/n^2$ when $E = \mathbb{N}[1, q]$ to $n(n - p)/n^2$ when $E = \mathbb{N}[n - p + 1, n - p + q]$. Hence for all r satisfying (3.14) at least one subset E can be found for which $Q(A_3, A_1) = nr/n^2 = \gamma_{123}$.

The second case is the one where $p,\ q,\ r$ satisfy

$$nq \leq n(p + q + r - n) \leq n(q + r) - qr\,,$$

or, equivalently,

$$n(n-r) \leq np \leq n^2 - qr\,. \tag{3.16}$$

We now define

$$\begin{aligned} A_1 &= E \cup \mathbb{N}[2n+r+1, 3n]\,, \\ A_2 &= E^c \cup \mathbb{N}[2n-q+r+1, 2n+r]\,, \\ A_3 &= \mathbb{N}[n-q+r+1, 2n-q+r]\,, \end{aligned} \tag{3.17}$$

with E a subset with cardinality r of $\mathbb{N}[1, n-q+r]$ and $E^c = \mathbb{N}[1, n-q+r] \setminus E$. E^c has cardinality $(n-q)$. From (3.17) it immediately follows that $Q(A_2, A_3) = nq/n^2 = \beta_{123}$ and $Q(A_3, A_1) = nr/n^2 = \gamma_{123}$. Depending upon the choice of E we obtain that $Q(A_1, A_2)$ can vary from $n(n-r)/n^2$ when $E = \mathbb{N}[1, r]$ to $(n(n-r) + r(n-q))/n^2 = (n^2 - qr)/n^2$ when $E = \mathbb{N}[n-q+1, n-q+r]$. In particular, for all p satisfying (3.16) at least one subset E can be found for which $Q(A_1, A_2) = np/n^2 = \alpha_{123}$. ∎

Note that, as $T_{\mathbf{P}}$-transitivity implies dice-transitivity, a 3-dimensional $T_{\mathbf{P}}$-transitive probabilistic relation with rational elements can also be generated by the standard triplets constructed in Theorem 3.4.7 which are in general different from the standard triplets obtained from Proposition 3.4.4. The generated $T_{\mathbf{M}}$-transitive standard triplets, however, are the same for both constructions. A graphical representation of the two constructions for $T_{\mathbf{M}}$-transitive triplets, corresponding to (3.8) and (3.9), is shown in Figure 3.2.

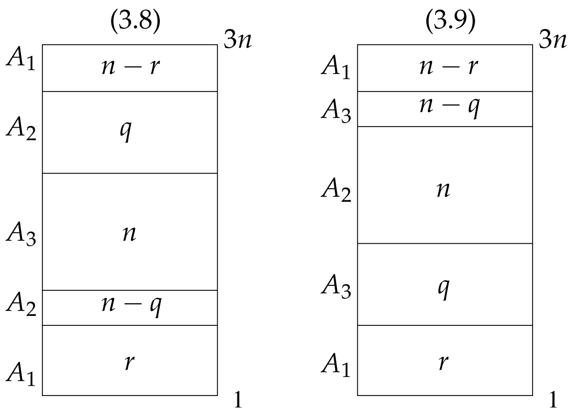


Figure 3.2: *Two constructions for $T_{\mathbf{M}}$-transitive triplets.*

3.5 Dice-transitivity revisited

3.5.1 The probabilistic sum as an upper bound function

The function $x + y - xy$ is generally known as the probabilistic sum. When comparing the upper bound functions for $T_{\mathbf{P}}$-transitivity and dice-transitivity, we see they are very similar. Indeed, they both can be seen as a probabilistic sum, upper bound function $U_{\mathbf{P}}$ uses as arguments α and β while upper bound function U_D uses β and γ. Once this is observed, an immediate question pops up with what type of relations the upper bound function $U_{PD}(\alpha, \beta, \gamma) = \alpha + \gamma - \alpha\gamma$, in other words the probabilistic sum of α and γ, is associated. In this section, we give a more general result concerning commutative quasi-copulas, which will also answer the above question.

We start by stating the following interesting fact about probabilistic relations.

PROPOSITION – 3.5.1: *For any probabilistic relation* $Q = [q_{ij}]$ *and for any* $[0,1]^2 \to [0,1]$ *mapping* $g \leq \min$*, it holds that*

$$(\forall\,(i,j,k))\,(q_{ik} \geq g(q_{ij}, q_{jk}) \vee q_{ki} \geq g(q_{kj}, q_{ji}))\,.$$

Proof:
Suppose, for some i, j, k it holds that $q_{ik} < g(q_{ij}, q_{jk})$. We will prove that this implies $q_{ki} > g(q_{kj}, q_{ji})$. We find $q_{ki} > g(q_{kj}, q_{ji}) \Leftrightarrow 1 - q_{ik} > g(1 - q_{jk}, 1 - q_{ij}) \Leftrightarrow q_{ik} < 1 - g(1 - q_{jk}, 1 - q_{ij})$. As $g \leq \min$, we have that $g(q_{ij}, q_{jk}) \leq 1 - g(1 - q_{jk}, 1 - q_{ij})$ and as $q_{ik} < g(q_{ij}, q_{jk})$ we indeed obtain the desired result. ∎

Note that the above proposition holds in particular for any quasi-copula g and any t-norm g. A graphical interpretation can be given by observing the following graphs, which we will call g-graphs: for any 3 elements a, b, c from a set of alternatives that generates a probabilistic relation $Q = [q_{ij}]$, consider a graph with nodes a, b and c. An arc from node i to j is drawn whenever it holds that $q_{ij} \geq g(q_{ik}, q_{kj})$, $i, j \in \{a, b, c\}$, $i \neq j$. We call such a graph (with 3 nodes) a g-graph corresponding to 3 alternatives. The above proposition then says that all g-graphs, with $g \leq \min$, obtained from m-dimensional probabilistic relations, have at least one arc between any two nodes.

Before continuing, we first prove a useful lemma concerning commutative quasi-copulas.

LEMMA – 3.5.2: *When* g *is a commutative quasi-copula, it holds that* $\alpha + \beta - 1 + g(\pi_1(1-\alpha, 1-\beta)) \leq \alpha + \gamma - 1 + g(\pi_2(1-\alpha, 1-\gamma)) \leq \beta + \gamma - 1 + g(\pi_3(1-\beta, 1-\gamma)) \leq \alpha + \beta - g(\pi_4(\alpha, \beta)) \leq \alpha + \gamma - g(\pi_5(\alpha, \gamma)) \leq \beta + \gamma - g(\pi_6(\beta, \gamma))$*. Here,* $\pi_i(x, y)$ *can be substituted by* (x, y) *or* (y, x)*.*

Proof:
As g is a quasi-copula, it holds that g is 1-Lipschitz and increasing, which implies that $g(x_1, y_1) - g(x_2, y_2) \leq x_1 - x_2 + y_1 - y_2$, for any $0 \leq x_2 \leq x_1 \leq 1$ and $0 \leq y_2 \leq y_1 \leq 1$. As g is commutative, we only need to prove the statements for one specific choice of π_i, $1 \leq i \leq 6$. Let $x_1 = x_2 = x$, then $g(x, y_1) - g(x, y_2) \leq y_1 - y_2$. Choosing $(x, y_1, y_2) = (1-\alpha, 1-\beta, 1-\gamma)$ and $(x, y_1, y_2) = (1-\gamma, 1-\alpha, 1-\beta)$ proves the first two inequalities while choosing $(x, y_1, y_2) = (\alpha, \gamma, \beta)$ and $(x, y_1, y_2) = (\gamma, \beta, \alpha)$ proves the last two

inequalities. The third inequality follows from the fact that g is a quasi-copula and therefore $g \leq \min$: $\beta+\gamma-1+g(1-\beta,1-\gamma) \leq \beta \leq \alpha+\beta-g(\alpha,\beta)$. ∎

We start now by showing that cycle-transitivity w.r.t. $U(\alpha,\beta,\gamma)=\beta+\gamma-g(\beta,\gamma)$ is equivalent to the condition that all the g-graphs have at least 4 arcs.

PROPOSITION – 3.5.3: *A probabilistic relation $Q=[q_{ij}]$ is cycle-transitive w.r.t. the upper bound function $U(\alpha,\beta,\gamma)=\beta+\gamma-g(\beta,\gamma)$, with g a commutative quasi-copula, if and only if*

$$\begin{gathered}(\forall\,(i,j,k))\,(\exists\,(i',j',k')=\pi(i,j,k))\\ (q_{i'k'} \geq g(q_{i'j'},q_{j'k'}) \wedge q_{k'i'} \geq g(q_{k'j'},q_{j'i'}))\,,\end{gathered} \tag{3.18}$$

where $\pi(i,j,k)$ represents a permutation of (i,j,k).

Proof:
For any (i,j,k), condition (3.18) is equivalent to the following condition, for an arbitrary loop direction:

$$\begin{array}{l}\alpha+\beta+g(1-\alpha,1-\beta)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\beta-g(\alpha,\beta) \ \vee\\ \alpha+\gamma+g(1-\alpha,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\gamma-g(\alpha,\gamma) \ \vee\\ \beta+\gamma+g(1-\beta,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \beta+\gamma-g(\beta,\gamma)\,.\end{array}$$

Due to Lemma 3.5.2, the above condition is equivalent to $\alpha+\beta+g(1-\alpha,1-\beta)-1 \leq \alpha+\beta+\gamma-1 \leq \beta+\gamma-g(\beta,\gamma)$. ∎

The proposition below shows that cycle-transitivity w.r.t. the upper bound function $U(\alpha,\beta,\gamma)=\alpha+\gamma-g(\alpha,\gamma)$ is equivalent to the condition that all the g-graphs have at least 5 arcs.

PROPOSITION – 3.5.4: *A probabilistic relation $Q=[q_{ij}]$ is cycle-transitive w.r.t. the upper bound function $U(\alpha,\beta,\gamma)=\alpha+\gamma-g(\alpha,\gamma)$, with g a commutative quasi-copula, if and only if*

$$\begin{gathered}(\forall\,(i,j,k))\,(\exists\,(i',j',k')=\pi(i,j,k))\\ \left(\begin{array}{l} q_{i'k'} \geq g(q_{i'j'},q_{j'k'}) \wedge q_{k'i'} \geq g(q_{k'j'},q_{j'i'}) \ \wedge\\ q_{i'j'} \geq g(q_{i'k'},q_{k'j'}) \wedge q_{j'i'} \geq g(q_{j'k'},q_{k'i'})\end{array}\right)\,,\end{gathered} \tag{3.19}$$

where $\pi(i,j,k)$ represents a permutation of (i,j,k).

Proof:
Due to Lemma 3.5.2, for any (i,j,k), condition (3.19) is equivalent to the following condition, for an arbitrary loop direction:

$$\begin{array}{l}\alpha+\gamma+g(1-\alpha,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\beta-g(\alpha,\beta) \ \vee\\ \beta+\gamma+g(1-\beta,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\beta-g(\alpha,\beta) \ \vee\\ \beta+\gamma+g(1-\beta,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\gamma-g(\alpha,\gamma)\,.\end{array}$$

Finally, again due to Lemma 3.5.2, the above condition is equivalent to

$$\alpha+\gamma+g(1-\alpha,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\gamma-g(\alpha,\gamma)\,.$$

∎

To conclude this subsection, we show that cycle-transitivity w.r.t. an upper bound function $U(\alpha,\beta,\gamma) = \alpha+\beta-g(\alpha,\beta)$ is equivalent to the condition that all the g-graphs have 6 arcs.

PROPOSITION – 3.5.5: *A probabilistic relation $Q = [q_{ij}]$ is cycle-transitive w.r.t. the upper bound function $U(\alpha,\beta,\gamma) = \alpha+\beta-g(\alpha,\beta)$, with g a commutative quasi-copula, if and only if*

$$(\forall\,(i,j,k))\,(q_{ij} \geq g(q_{ik},q_{kj}))\,. \tag{3.20}$$

Proof:
It follows directly from Lemma 3.5.2 that (3.20) is equivalent to $\beta+\gamma+g(1-\beta,1-\gamma)-1 \leq \alpha+\beta+\gamma-1 \leq \alpha+\beta-g(\alpha,\beta)$. ■

As a final note, we wish to remark that the framework of cycle-transitivity allows for very elegant equivalent conditions to (3.18), (3.19) and (3.20).

3.5.2 Constructing dice

As follows from the constructional proof of Theorem 3.4.7, more specifically from (3.15) and (3.17), to generate any 3-dimensional dice-transitive relation, with rational elements, by means of a standard triplet, the cardinality of one of the dice can always be chosen to be one. In this subsection we show that the construction can be simplified even more. To achieve this further simplification, we need to introduce the notion of blocks of integers corresponding to a given standard triplet. Consider a standard triplet for which the three dice have n_1, resp. n_2, resp. n_3 faces. Partition $\mathbb{N}[1,n_1+n_2+n_3]$ into the maximal sets N_i for which it holds that they contain successive integers that all appear on the same dice. It is obvious that the fewer the number of blocks are for a given standard triplet, the simpler the triplet. Calculating the generated probabilistic relation, for example, can be done faster if there are less blocks. Figure 3.2, for example, shows two triplets, each being composed out of 5 blocks. For 3-dimensional $T_{\mathbf{M}}$-transitive relations it therefore holds that they can be generated by a triplet consisting of at most 5 blocks.

EXAMPLE – 3.5.6:
Consider as another example the standard triplet consisting of

$$A_1 = \{2,3,4,8,9,12,13,14,15,16\}\,,\quad A_2 = \{1,6,7\}\,,\quad A_3 = \{5,10,11\}\,.$$

The above triplet thus consists of the following seven blocks:
$\mathbb{N}[1,1], \mathbb{N}[2,4], \mathbb{N}[5,5], \mathbb{N}[6,7], \mathbb{N}[8,9], \mathbb{N}[10,11], \mathbb{N}[12,16]$. ◀

Using the above definition, we are now able to state the following proposition.

PROPOSITION – 3.5.7: *All 3-dimensional dice-transitive probabilistic relations with rational elements can be generated by a standard triplet consisting of at most seven blocks. Furthermore, the cardinality of one of the dice can be chosen to be 1.*

Proof:
Assume we have been given a dice-transitive graph with 3 nodes and the weights of a certain loop given by

$$\alpha = \frac{p}{n} \leq \beta = \frac{q}{n} \leq \gamma = \frac{r}{n}\,. \tag{3.21}$$

Note that the arcs don't necessarily carry those weights in that specific order. For the construction we must have p, q, r, n such that n is a divisor of qr. These conditions are always satisfied when multiplying the 4 given numbers by n. We thus obtain: $p' = pn$, $q' = qn$, $r' = rn$ and $n' = nn$. It now holds that $p'/n' = p/n$, $q'/n' = q/n$, $r'/n' = r/n$ and $q'r'/n' = qr \in \mathbb{N}$. For the remainder of the proof we will drop the accents and assume that n is a divisor of qr. Furthermore, given α, β, γ, we can assume that

$$\beta\gamma \leq \alpha + \beta + \gamma - 1 \leq \beta + \gamma - \beta\gamma, \tag{3.22}$$

because if this property does not hold it will hold for the other loop direction where $\gamma' = 1 - \alpha$, $\beta' = 1 - \beta$, $\alpha' = 1 - \gamma$ and we can use these values instead (dropping the accents).

For simplicity, let us assume that we can relabel the indices such that $q_{23} = \beta$ and $q_{31} = \gamma$. If that's impossible, we can relabel the indices such that $q_{23} = \gamma$ and $q_{31} = \beta$ and the proof is then completely analogous. We will prove that for any $\alpha = q_{12}$ with a value between

$$\frac{(n-r)(n-q)}{n^2} = (1-\beta)(1-\gamma) \tag{3.23}$$

and

$$\min\left(\frac{(n-r)n + r(n-q)}{n^2}, \beta\right) = \min(1 - \beta\gamma, \beta)\,, \tag{3.24}$$

we can construct a standard triplet consisting of at most 7 blocks. Note that the condition (3.23) $\leq \alpha \leq$ (3.24) is equivalent to (3.22). For the upper bound of α we must make sure it is not higher than β, hence the minimum operator in (3.24).

We note that when α equals the lower bound (3.23) or the upper bound (3.24), it holds that $\alpha n \in \mathbb{N}$ (because we can assume $qr/n \in \mathbb{N}$), so both these bounds already represent a possible value for $p/n = \alpha$ (the minimal and maximal value).

All values for $\alpha = p/n$ located between the lower bound (3.23) and the bound

$$(1-\beta)(1-\gamma) + \max(\min(\beta, 1-\beta), \min(\gamma, 1-\gamma)), \tag{3.25}$$

can be generated by a standard triplet with no more than 7 blocks. Indeed, suppose that $\alpha = (3.23) + l/n$, with $0 \leq l \leq \min(r, n-r)$. Then the standard triplet below generates this dice-transitive relation.

$$\begin{aligned}
A_1 &= \mathbb{N}[1, r-l] \cup \mathbb{N}[n+r-q-l+1, n+r-q] \cup \\
&\quad\ \mathbb{N}[n+r-q+2, 2n-q+1-l] \cup \mathbb{N}[2n+2-l, 2n+1]\,, \\
A_2 &= \mathbb{N}[r+1-l, n+r-q-l] \cup \mathbb{N}[2n-q+2-l, 2n+1-l]\,, \\
A_3 &= \{n+r-q+1\}\,.
\end{aligned}$$

Secondly, suppose that $\alpha = (3.23) + l/n$, with $0 \leq l \leq \min(q, n-q)$. Then the standard triplet below generates the dice-transitive relation.

$$\begin{aligned}
A_1 &= \mathbb{N}[l+1, r+l] \cup \mathbb{N}[n+r-q+2+l, 2n-q+l+1]\,,\\
A_2 &= \mathbb{N}[1, l] \cup \mathbb{N}[r+l+1, n+r-q] \cup\\
&\quad\ \mathbb{N}[n+r-q+2, n+r-q+1+l] \cup \mathbb{N}[2n-q+l+2, 2n+1]\,,\\
A_3 &= \{n+r-q+1\}\,.
\end{aligned}$$

Similarly, all values for $\alpha = p/n$ located between the upper bound (3.24) and the (lower) bound

$$1 - \beta\gamma - \max(\min(\beta, 1-\beta), \min(\gamma, 1-\gamma)), \tag{3.26}$$

can be generated by a standard triplet consisting of at most 7 blocks. Indeed, suppose that $\alpha = 1 - \beta\gamma - l/n$, with $0 \leq l \leq \min(r, n-r)$. Then the standard triplet below generates the dice-transitive relation.

$$\begin{aligned}
A_1 &= \mathbb{N}[1, l] \cup \mathbb{N}[n-q+l+1, n+r-q] \cup\\
&\quad\ \mathbb{N}[n+r-q+2, n+r-q+l+1] \cup \mathbb{N}[n+r+l+2, 2n+1]\,,\\
A_2 &= \mathbb{N}[l+1, n-q+l] \cup \mathbb{N}[n+r-q+l+2, n+r+l+1]\,,\\
A_3 &= \{n+r-q+1\}\,.
\end{aligned}$$

Finally, suppose that $\alpha = 1 - \beta\gamma - l/n$, with $0 \leq l \leq \min(q, n-q)$. Then the standard triplet below generates the dice-transitive relation.

$$\begin{aligned}
A_1 &= \mathbb{N}[n-q-l+1, n+r-q-l] \cup \mathbb{N}[n+r-l+2, 2n-l+1]\,,\\
A_2 &= \mathbb{N}[1, n-q-l] \cup \mathbb{N}[n+r-q-l+1, n+r-q] \cup\\
&\quad\ \mathbb{N}[n+r-q+2, n+r-l+1] \cup \mathbb{N}[2n-l+2, 2n+1]\,,\\
A_3 &= \{n+r-q+1\}\,.
\end{aligned}$$

Note that all the above standard triplets, constructed for each of the four cases, consist of at most 7 blocks. It remains to be proven that all valid values for $\alpha = p/n$ satisfying (3.22) are within the union of the two intervals. We divide all possibilities into three cases.

Case 1: $\beta,\ \gamma \leq 1/2$.

The border (3.25) now becomes $(1-\beta)(1-\gamma)+\gamma$. As $\alpha \leq \gamma \leq (1-\beta)(1-\gamma)+\gamma$, α is always situated between the borders (3.23) and (3.25).

Case 2: $\beta,\ \gamma \geq 1/2$.

For this case, we will prove that $[(3.23), (3.24)] = [(3.23), (3.25)] \cup [(3.26), (3.24)]$, which implies that the construction is possible for all valid values of α. We must therefore prove that (3.25) $\geq$ (3.26). This is equivalent to

$$(1-\beta)(1-\gamma) + (1-\beta) \geq 1 - \beta\gamma - (1-\beta)\,,$$

which is equivalent to

$$-2 + 3\beta + \gamma(1-2\beta) \leq 0\,.$$

We now determine an upper bound for the left-hand side of the above inequality. If we assume γ to be a constant between 1/2 and 1 then we see that this function of β is strictly increasing. To obtain the maximum value of the left-hand side, we therefore always take the highest possible value for β, which is clearly given by $\beta = \gamma$. We now obtain the following function in γ which we want to maximize:

$$-\gamma^2 + 2\gamma - 1 = -(1-\gamma)^2 .$$

As the above function is always smaller or equal to zero, it indeed holds that (3.25) $\geq$ (3.26).

Case 3: $\beta < 1/2$, $\gamma > 1/2$.

Case 3.1: $\beta \geq 1 - \gamma$.

As $\alpha \leq \beta \leq (1-\beta)(1-\gamma) + \beta$, we obtain that α is always situated between the borders (3.23) and (3.25).

Case 3.1: $1 - \gamma > \beta$.

Because $\beta < 1 - \gamma$ we have that $\beta < (1-\beta)(1-\gamma) + 1 - \gamma$ and this again implies that α is situated between the borders (3.23) and (3.25). ∎

3.6 Towards higher dimensions

In [75], besides the utility model, which yields probabilistic relations that have strong transitivity properties, also the so-called multidimensional model is discussed. Moreover, it has been shown that the probabilistic relations generated by this multidimensional model are $T_{\mathbf{L}}$-transitive, and conversely, all $T_{\mathbf{L}}$-transitive probabilistic relations on a universe of dimension $n \leq 5$ can be generated by a multidimensional model [10, 76]. By analogy, as far as our model is concerned, the question arises whether the reverse property, which has been proven in Theorem 3.4.7 to hold for 3-dimensional probabilistic relations with rational elements, extends to higher-dimensional probabilistic relations. The question must be answered in negative sense, as follows from the next theorem.

THEOREM – 3.6.1: *Not all 4-dimensional dice-transitive probabilistic relations (with rational elements) can be generated by a dice model.*

Proof:

We will construct a set of graphs of which the associated probabilistic relation is dice-transitive but for which there does not exist a collection of dice (A_1, A_2, A_3, A_4) that generates it. We will use the graph of Figure 3.3, which shows explicitly that $Q(A_1, A_3) = e = 0$ and $Q(A_2, A_4) = f = 0$. Obviously, it holds that $a, b, c, d \in [0, 1]$. In this graph there are four subgraphs with 3 nodes. The condition of dice-transitivity has to hold for each subgraph. We therefore have the following four conditions that must hold:

$$\begin{cases} 0 \leq d - a \leq 1 - a(1-d) \ , \text{ for triplet } (A_1, A_2, A_4) , \\ 0 \leq d - c \leq 1 - c(1-d) \ , \text{ for triplet } (A_1, A_3, A_4) , \\ 0 \leq c - b \leq 1 - b(1-c) \ , \text{ for triplet } (A_2, A_4, A_3) , \\ 0 \leq a - b \leq 1 - b(1-a) \ , \text{ for triplet } (A_2, A_1, A_3) , \end{cases}$$

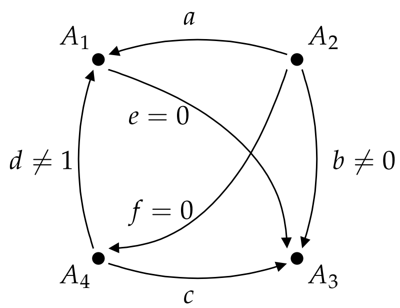


Figure 3.3: *Dice-transitive probabilistic relations that cannot be generated by a dice model.*

which is equivalent to

$$b \leq c \leq d \ \wedge \ b \leq a \leq d \,. \tag{3.27}$$

Note that these conditions can easily be satisfied. We now prove that, when $e = 0$, $f = 0$, when the conditions (3.27) are fulfilled and when

$$b \neq 0 \ \wedge \ d \neq 1 \,, \tag{3.28}$$

we have examples of graphs that are dice-transitive but that cannot be generated by a collection of 4 dice.

Let us assume that such a collection (A_1, A_2, A_3, A_4) does exist and let $a_1 = \max A_1$ and $a_2 = \max A_2$. We have two cases. In the first case we have $a_1 > a_2$ from which it follows that $b = 0$. In the second case we have $a_1 \leq a_2$ from which it follows that $d = 1$. In the first case we used the fact that $e = 0$ and in the second case that $f = 0$. These two cases represent all possible situations and (3.28) does not hold in either case. Therefore, there exist no standard quartets that correspond to the dice-transitive graph from Figure 3.3 having the following properties:

$$b \leq c \leq d \,,\ b \leq a \leq d \,,\ b \neq 0 \,,\ d \neq 1 \,,\ e = 0 \,,\ f = 0 \,. \tag{3.29}$$

Again, the conditions (3.29) can easily be satisfied. ■

Numerous attempts have been made to obtain the exact characterization of the transitivity of 4-dimensional dice models. Most of these attempts concerned finding more restrictive upper bounds such that all known 6-tuples of probabilities that were known to be generated by 4-dimensional dice models satisfied the imposed conditions. These attempts were mostly unsatisfactory and we therefore will not report on them.

The result from Proposition 3.5.7, however, seems to suggest a general rule. First note that any 2-dimensional dice-transitive relation with rational elements can be generated by two dice of which one dice only has one face. The general rule could be that any m-dimensional probabilistic relation generated by an m-dimensional dice model can always be generated by an m-dimensional standard collection composed of at most $2^m - 1$ blocks and for which one dice can be chosen to have only one face. However, the proposition below shows that this generalization cannot be done.

PROPOSITION – 3.6.2: *Not all 4-dimensional dice-transitive relations with rational elements can be generated by a dice model of which at least one dice has only one face.*

Proof:
We will show that the dice-transitive probabilistic relation $Q = [q_{ij}]$ with elements $q_{12} = 7/9$, $q_{13} = 1/2$, $q_{14} = 8/9$, $q_{23} = 0$, $q_{24} = 4/9$ and $q_{34} = 3/4$ can be generated by a 4-dimensional dice model but cannot be generated by a dice model in which one dice has only one face.

First of all, it is easily verified that the standard quartet (A_1, A_2, A_3, A_4), with

$$A_1 = \{4, 11, 12\}\,, A_2 = \{1, 5, 6\}\,, A_3 = \{7, 8, 9, 13\}\,, A_4 = \{2, 3, 10\}\,,$$

generates the above probabilistic relation and that the above four dice each consists of more than 1 block. We now consider another standard quartet (A'_1, A'_2, A'_3, A'_4) that generates the same probabilistic relation and show for each dice that it must consist of more than one block and, consequently, have more than one face.
Case 1: $\#A'_1 = 1$.
It must then hold that $q_{23} \geq q_{21}q_{13}$, which is equivalent to $0 \geq 1/9$.
Case 2: $\#A'_2 = 1$.
It must then hold that $q_{41} \geq q_{42}q_{21}$, which is equivalent to $1/9 \geq 10/81$.
Case 3: $\#A'_3 = 1$.
It must then hold that $q_{41} \geq q_{43}q_{31}$, which is equivalent to $1/9 \geq 1/8$.
Case 4: $\#A'_4 = 1$.
It must then hold that $q_{23} \geq q_{24}q_{43}$, which is equivalent to $0 \geq 1/9$. ■

We can, however, state sufficient conditions on a 4-dimensional probabilistic relation such that it can be generated by a 4-dimensional dice model. We will now show that any 4-dimensional $T_{\mathbf{M}}$-transitive probabilistic relation with rational elements can be generated by a dice model. We start by showing that in all $T_{\mathbf{M}}$-transitive digraphs with 4 nodes, corresponding to a 4-dimensional probabilistic relation, there exists a weighted arc such that its weight and the weight of the arc in the reversed direction are the middle weight in the loops of all triplets that contain this weighted arc (or the arc in the reversed direction).

LEMMA – 3.6.3: *For any 4-dimensional $T_{\mathbf{M}}$-transitive probabilistic relation $Q = [q_{ij}]$, there exists a permutation $(p_1, p_2, p_3, p_4) = \pi(1, 2, 3, 4)$ such that*

$$\begin{aligned} q_{p_1p_2} &= \mathrm{median}(q_{p_2p_3}, q_{p_3p_1}, q_{p_1p_2}) \quad \wedge \\ q_{p_1p_2} &= \mathrm{median}(q_{p_2p_4}, q_{p_4p_1}, q_{p_1p_2})\,. \end{aligned} \tag{3.30}$$

Proof:
The proof consists of considering all possible situations. In each possible situation, we will give values for p_1 and p_2 such that (3.30) is satisfied. Without any loss of generality we can assume that $q_{12} = \beta_{123}$. This implies that $q_{23} = 1 - q_{31}$.

Case 1: $q_{12} = \beta_{124}$.
Choose $p_1 = 1$, $p_2 = 2$.

Case 2: $q_{24} = \beta_{124}$ — implying $q_{12} = 1 - q_{41}$.

Case 2.1: $q_{24} = \beta_{324}$.
Choose $p_1 = 2$, $p_2 = 4$.

Case 2.2: $q_{32} = \beta_{324}$ — implying $q_{24} = 1 - q_{43}$.

Case 2.2.1: $q_{14} = \beta_{143}$ — implying $q_{43} = 1 - q_{31}$.
As $\beta_{124} = q_{24} = q_{34} = q_{31} = q_{32} = \beta_{324}$, we choose $p_1 = 2$, $p_2 = 4$.

Case 2.2.2: $q_{43} = \beta_{143}$ — implying $q_{31} = 1 - q_{14}$.
As $\beta_{324} = q_{32} = q_{31} = q_{41} = q_{21} = \beta_{321}$, we choose $p_1 = 2$, $p_2 = 3$.

Case 2.2.3: $q_{31} = \beta_{143}$ — implying $q_{14} = 1 - q_{43}$.
As $\beta_{123} = q_{12} = q_{14} = q_{34} = q_{24} = \beta_{124}$, we choose $p_1 = 1$, $p_2 = 2$.

Case 2.3: $q_{43} = \beta_{324}$ — implying $q_{32} = 1 - q_{24}$.

Case 2.3.1: $q_{43} = \beta_{143}$.
Choose $p_1 = 3$, $p_2 = 4$.

Case 2.3.2: $q_{14} = \beta_{143}$ — implying $q_{43} = 1 - q_{31}$.
As $\beta_{124} = q_{24} = q_{23} = q_{13} = q_{43} = \beta_{324}$, we choose $p_1 = 2$, $p_2 = 4$.

Case 2.3.3: $q_{31} = \beta_{143}$ — implying $q_{14} = 1 - q_{43}$.
In this case we obtain the following 4 equalities: $q_{23} = 1 - q_{31}$, $q_{12} = 1 - q_{41}$, $q_{32} = 1 - q_{24}$, $q_{14} = 1 - q_{43}$. They imply that there exist $a, b \in [0,1]$ such that $q_{12} = a$, $q_{23} = b$, $q_{34} = a$, $q_{14} = a$, $q_{13} = b$ and $q_{24} = b$, with $\min(a, 1-a) \le b \le \max(a, 1-a)$ and $\min(b, 1-b) \le a \le \max(b, 1-b)$. For (3.30) not to be satisfied, the above inequalities must be strict inequalities:

$$\begin{aligned} a &> \min(b, 1-b)\,, & b &> \min(a, 1-a)\,, \\ 1-a &> \min(1-b, b)\,, & 1-b &> \min(1-a, a)\,. \end{aligned}$$

Firstly, assume $1 - b \le b$. The aforementioned strict inequalities then imply $1 - b < a \wedge 1 - b < 1 - a$, which contradicts $1 - b > \min(a, 1-a)$. Lastly, assume $1 - b > b$. We then obtain $b < a \wedge b < 1 - a$, which in turn contradicts $b > \min(a, 1-a)$.

Case 3: $q_{41} = \beta_{124}$ — implying $q_{12} = 1 - q_{24}$.

Case 3.1: $q_{14} = \beta_{143}$.
Choose $p_1 = 1$, $p_2 = 4$.

Case 3.2: $q_{43} = \beta_{143}$ — implying $q_{14} = 1 - q_{31}$.

Case 3.2.1: $q_{43} = \beta_{324}$.
Choose $p_1 = 3$, $p_2 = 4$.

Case 3.2.2: $q_{24} = \beta_{324}$ — implying $q_{32} = 1 - q_{43}$.
As $\beta_{124} = q_{41} = q_{31} = q_{32} = q_{34} = \beta_{134}$, we choose $p_1 = 1$, $p_2 = 4$.

Case 3.2.3: $q_{32} = \beta_{324}$ — implying $q_{24} = 1 - q_{43}$.
This case is completely analogous to Case 2.3.3.
Case 3.3: $q_{31} = \beta_{143}$ — implying $q_{14} = 1 - q_{43}$.
Case 3.3.1: $q_{43} = \beta_{324}$ — implying $q_{32} = 1 - q_{24}$.
As $\beta_{143} = q_{31} = q_{32} = q_{42} = q_{12} = \beta_{123}$, we choose $p_1 = 1$, $p_2 = 3$.
Case 3.3.2: $q_{24} = \beta_{324}$ — implying $q_{32} = 1 - q_{43}$.
As $\beta_{124} = q_{41} = q_{43} = q_{23} = q_{13} = \beta_{134}$, we choose $p_1 = 1$, $p_2 = 4$.
Case 3.3.3: $q_{32} = \beta_{324}$ — implying $q_{24} = 1 - q_{43}$.
As $\beta_{123} = q_{12} = q_{42} = q_{43} = q_{41} = \beta_{124}$, we choose $p_1 = 1$, $p_2 = 2$.

■

PROPOSITION – 3.6.4: *Any 4-dimensional $T_{\mathbf{M}}$-transitive probabilistic relation with rational elements $Q = [q_{ij}]$ can be generated by a standard quartet.*

Proof:
Lemma 3.6.3 implies that for any 4-dimensional probabilistic $T_{\mathbf{M}}$-transitive relation $Q = [q_{ij}]$, the indices i, j can be permuted such that $q_{12} = \beta_{123}$ and $q_{12} = \beta_{124}$. We will refer to q_{12} as β. Furthermore, the indices can be chosen such that $q_{24} \geq q_{41}$. We will refer to q_{24} as γ and therefore to q_{41} as $1 - \gamma$. The weights q_{23} and q_{31} will be referred to as γ' and $1 - \gamma'$, respectively. Note that we cannot assume, without loss of generality, that $\gamma' \geq 1 - \gamma'$, however we will not need such an assumption. As the relation has rational elements, we can, as before, find a common denominator n for the values q_{ij}. Let $\beta = q/n$, $\gamma = r/n$ and $\gamma' = r'/n$.

We now consider the following standard quartets:

$$\begin{aligned} A_1 &= \mathbb{N}[r + r' + q + 1, r + r' + q + n]\,, \\ A_2 &= \mathbb{N}[r + r' + 1, r + r' + q] \cup \\ &\quad\ \mathbb{N}[r + r' + q + n + 1, 2n + r + r']\,, \\ A_3 &= E_1 \cup E_2\,, \\ A_4 &= E_1^c \cup E_2^c\,. \end{aligned} \tag{3.31}$$

Here, E_1 is an r'-dimensional subset of $\mathbb{N}[1, r + r']$ and $E_1^c = \mathbb{N}[1, r + r'] \setminus E_1$. Note that E_1^c is r-dimensional. On the other hand, E_2 is an $(n - r')$-dimensional subset of $\mathbb{N}[2n + r + r' + 1, 4n]$ and $E_2^c = \mathbb{N}[2n + r + r' + 1, 4n] \setminus E_2$. Hence E_2^c is $(n - r)$-dimensional.

For the probabilistic relations generated from the quartets (3.31), it is clear that $q_{12} = q/n = \beta$, $q_{23} = q_{13} = r'/n = \gamma'$ and $q_{24} = q_{14} = r/n = \gamma$. The only remaining value is q_{43}. Depending upon the choice of E_1 and E_2, the possible values for q_{43} are those for which the double inequality below holds:

$$(n - r)r' \leq n^2 q_{43} \leq n(n - r) + rr'\,. \tag{3.32}$$

We now distinguish the three possible cases for β_{243} and verify that all possible values of q_{43} satisfy the double inequality (3.32).

Case 1: $q_{43} = \beta_{243}$.
This implies $q_{32} = 1 - q_{24}$ which is equivalent to $\gamma = \gamma'$ or, equivalently, $r = r'$. Thus $1 - \gamma \leq q_{43} \leq \gamma$, or equivalently $n(n-r) \leq n^2 q_{43} \leq nr$. As r and r' are smaller or equal to n we have that all values q_{43} for this case satisfy (3.32). Indeed, $n(n-r) + r^2 \geq nr \Leftrightarrow n(n-r) \geq r(n-r)$, which is always satisfied. Similarly, $(n-r)r \leq (n-r)n$ is also always satisfied.
Case 2: $q_{32} = \beta_{243}$.
This implies $q_{43} = 1 - \gamma = (n-r)/n$ and (3.32) is therefore satisfied.
Case 3: $q_{24} = \beta_{243}$.
This implies $q_{43} = \gamma' = r'/n$ and (3.32) is again satisfied. ■

The great tragedy of science - the slaying of a beautiful hypothesis by an ugly fact.
— THOMAS HUXLEY

4 Comparing Independent Random Variables

The discrete dice model, which was discussed in the previous chapter, essentially amounts to the pairwise comparison of a collection of independent discrete random variables that are uniformly distributed on finite integer multisets. This pairwise comparison results in a probabilistic relation that exhibits a particular type of transitivity, called dice-transitivity. In this chapter, the discrete dice model is generalized with the purpose of pairwisely comparing independent discrete or continuous random variables with arbitrary probability distributions. It is shown that the probabilistic relation generated by a collection of arbitrary independent random variables is still dice-transitive. Interestingly, this probabilistic relation can be seen as a graded alternative to the concept of stochastic dominance [5, 37, 59, 60, 62]. Furthermore, when the marginal distributions of the random variables belong to the same parametric family of distributions, the probabilistic relation exhibits interesting types of isostochastic transitivity, such as multiplicative transitivity. Finally, the probabilistic relation generated by a collection of independent normal random variables is proven to be moderately stochastic transitive.

The outline of this chapter is as follows. In Section 4.1 we introduce the concept of a generalized discrete or continuous dice model and show that its probabilistic relation can, for the case of independent r.v., be interpreted as a graded alternative to the notion of stochastic dominance. Section 4.2 is concerned with the characterization of the type of transitivity exhibited by probabilistic relations of generalized dice models. The remaining sections are then devoted to the study of the influence particular choices of independent random variables have on the transitivity of the generated probabilistic relation. In Section 4.3 we focus on random variables with shifted distributions, in Section 4.4 on random variables with distributions taken from certain parametric families of distributions, and in Section 4.5 special attention is paid to the case of normal random variables. Most results from this chapter can be found in [18, 23, 25, 28].

4.1 Generalized dice models

Clearly, Definition 3.2.2 of the probabilistic relation Q of a discrete dice model can be immediately extended to compare arbitrary random variables. Indeed, any collection of r.v. $(X_1, X_2, \ldots, X_m)$ can, by means of the pairwise comparison of its components, serve as a source for generating an m-dimensional probabilistic relation.

DEFINITION – 4.1.1: *Let $(X_1, X_2, \ldots, X_m)$ be a collection of m random variables, then the relation $Q = [q_{ij}]$ defined by*

$$q_{ij} = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\,\mathrm{Prob}\{X_i = X_j\} \tag{4.1}$$

is a probabilistic relation. The collection of random variables is called a generalized dice model.

The definition of $Q = [q_{ij}]$ implies that when the r.v. are coupled by absolutely continuous copulas, the elements q_{ij} can be computed from the bivariate joint cumulative distribution functions (c.d.f.) F_{X_i,X_j} as follows

$$q_{ij} = \int_{x>y} dF_{X_i,X_j}(x,y) + \frac{1}{2}\int_{x=y} dF_{X_i,X_j}(x,y)\,. \tag{4.2}$$

In this chapter, however, we will consider independent random variables only. The underlying copula is therefore the (absolutely continuous) product-copula and therefore bivariate distributions can always be factorized into their univariate marginal distributions. The case when the random variables are differently coupled will be considered in Chapter 6. If we want to further simplify (4.2) for the case of independent r.v., it is appropriate to distinguish between the following two cases.

DEFINITION – 4.1.2: *Let* $(X_1, X_2, \ldots, X_m)$ *be a collection of independent discrete random variables, then the relation* $Q = [q_{ij}]$ *defined by*

$$q_{ij} = \sum_{k>l} p_{X_i}(k) p_{X_j}(l) + \frac{1}{2}\sum_{k} p_{X_i}(k) p_{X_j}(k)\,, \tag{4.3}$$

with p_{X_i} *the marginal probability mass function of* X_i*, is a probabilistic relation. The collection of the discrete random variables is called a generalized discrete dice model (with independent random variables).*

DEFINITION – 4.1.3: *Let* $(X_1, X_2, \ldots, X_m)$ *be a collection of independent continuous random variables, then the relation* $Q = [q_{ij}]$ *defined by*

$$q_{ij} = \int_{-\infty}^{+\infty} f_{X_i}(x) \left(\int_{-\infty}^{x} f_{X_j}(y)\,dy\right) dx\,, \tag{4.4}$$

with f_{X_i} *the marginal probability density function of* X_i*, is a probabilistic relation. The collection of continuous random variables is called a generalized continuous dice model (with independent random variables).*

Note that in the transition from the discrete to the continuous case, the second contribution to q_{ij} in (4.2) has disappeared in (4.4), since in the latter case $\mathrm{Prob}\{X_i = X_j\} = 0$. Of course, the information contained in the probabilistic relation is much richer than if for the pairwise comparison of X_i and X_j we would have used, for instance, only their expected values $\mathrm{E}[X_i]$ and $\mathrm{E}[X_j]$.

In the discussion of generalized dice models, we will maintain the terminology related to the original discrete dice model. A collection of dice will be kept as a metaphor for a collection of random variables. Two dice X_i and X_j, taken from a collection of dice, are compared in terms of the quantity q_{ij} for which it holds that $q_{ij} = 1 - q_{ji}$. If $q_{ij} > 1/2$, we still say that dice X_i (statistically) wins from dice X_j (denoted as $X_i >_s X_j$), and if $q_{ij} = 1/2$, we say that both dice are (statistically) indifferent (denoted as $X_i =_s X_j$).

An alternative concept for comparing two random variables is that of stochastic dominance [59], which is particularly popular in financial mathematics.

DEFINITION – 4.1.4: *A random variable X_i with c.d.f. F_{X_i} stochastically dominates in first degree a random variable X_j with c.d.f. F_{X_j}, denoted as $X_i >_1 X_j$, if for all real t it holds that $F_{X_i}(t) \leq F_{X_j}(t)$, and the strict inequality holds for at least one t.*

The condition for first degree stochastic dominance is rather severe, as it requires that the graph of the function F_{X_i} lies beneath the graph of the function F_{X_j}. The need to relax this condition has led to other types of stochastic dominance, such as second degree and third degree stochastic dominance. We will not go into more details here, since we just want to emphasize the following relationship between first degree stochastic dominance and the winning probabilities of a dice model.

PROPOSITION – 4.1.5: *For any two independent random variables X_i and X_j it holds that $X_i >_1 X_j$ implies $X_i >_s X_j$.*

Proof:
We first give the proof for continuous random variables. Suppose that $X_i >_1 X_j$, implying $F_{X_i}(z) \leq F_{X_j}(z)$, for any $z \in \mathbb{R}$. Since F_{X_i} and F_{X_j} are right- or left-continuous functions, it holds that $F_{X_i}(z) < F_{X_j}(z)$, for any $z \in I \subseteq \mathbb{R}$, for at least one non-degenerated interval I. Therefore, we obtain

$$\text{Prob}\{X_i > X_j\} = \int_{-\infty}^{+\infty} f_{X_i}(x)F_{X_j}(x)dx > \int_{-\infty}^{+\infty} f_{X_i}(x)F_{X_i}(x)dx = \frac{1}{2}\,.$$

We now give the proof for the case of discrete random variables. Suppose $X_i >_1 X_j$. It is obvious that $\sum_{k>l} p_{X_i}(k)p_{X_i}(l) + 1/2\sum_{k=l}(p_{X_i}(k))^2 = 1/2$. As $X_i(k) \leq X_j(k)$, $k \in \mathbb{Z}$, with a strict inequality for at least one k, we obtain $\sum_{k>l} p_{X_i}(k)p_{X_j}(l) + 1/2\sum_{k=l} p_{X_i}(k)p_{X_j}(k) > 1/2$. ∎

The relation $>_s$ therefore generalizes first degree stochastic dominance $>_1$. As the probabilistic relation of a dice model is a graded version of the crisp relation $>_s$, we can therefore interpret this relation as a graded alternative to first degree stochastic dominance.

4.2 Transitivity of generalized dice models

One of the main results from Chapter 3 was the fact that the probabilistic relation generated by a dice model is dice-transitive. In this section, we prove that the probabilistic relation generated by any generalized dice model, whether discrete or continuous, with independent r.v. is at least dice-transitive. More precisely, we proceed in two distinct steps: firstly, the discrete dice model is generalized to cover the case of arbitrary discrete independent random variables, and, secondly, the generalization to arbitrary continuous independent random variables is considered.

THEOREM – 4.2.1: *The probabilistic relation of a generalized discrete dice model with independent random variables is dice-transitive.*

Proof:
First, we want to emphasize that the introduction of negative integers in the multisets of a discrete dice model does not alter the transitivity. Let X_k be a random variable of a generalized discrete dice model with independent r.v., and let $I_n = [-n, n]$, with $n > 0$. We now approximate the random variable X_k by a random variable $X_k^{(n)}$ which takes values in I_n with rational probabilities only, in such a way that:

$$
\begin{aligned}
&p_{X_k^{(n)}}(-n) \in \mathbb{Q} \ \wedge \ 0 \leq \mathrm{Prob}\{X_k \leq -n\} - p_{X_k^{(n)}}(-n) < \frac{1}{n^2}\,,\\
&p_{X_k^{(n)}}(j) \in \mathbb{Q} \ \wedge \ 0 \leq \mathrm{Prob}\{X_k = j\} - p_{X_k^{(n)}}(j) < \frac{1}{n^2}\,, \forall j \in I_m \setminus \{-n, n\}\,,\\
&p_{X_k^{(n)}}(n) = 1 - \sum_{i=-n}^{n-1} p_{X_k^{(n)}}(i)\,.
\end{aligned}
$$

It is clear that such an approximation always exists, since the set of rationals $\mathbb{Q}$ is dense in the set of reals $\mathbb{R}$. From the above inequalities, it also follows that

$$p_{X_k^{(n)}}(n) - \mathrm{Prob}\{X_k \geq n\} < \frac{2}{n}\,.$$

Since we can take n as large as we like, the generalized discrete dice model can be approximated with arbitrary precision, i.e. $\lim_{n\to+\infty} X_k^{(n)} = X_k$. For any $n \in \mathbb{N}$, the probabilistic relation generated by $X_k^{(n)}$ can also be generated by a discrete dice model in which the dice have a finite number of faces, each face containing one integer, and the probability of a particular face showing up in a random roll of the dice being for each face a rational number. Bringing all rational probabilities to a (least) common denominator, it suffices to duplicate, depending on the numerator values, each face a number of times in order to obtain an equivalent discrete dice model in the sense of Chapter 3 (see Section 3.3). As all such discrete dice models are dice-transitive it follows that all gereralized discrete dice models are dice-transitive. ∎

We now execute the second step mentioned before, by considering continuous dice models with independent r.v.

THEOREM – 4.2.2: *The probabilistic relation of a generalized continuous dice model with independent random variables is dice-transitive.*

Proof:
Let X_k be a random variable of a generalized continuous dice model with independent r.v. and with probability density function f_{X_k}. We partition $\mathbb{R}$ into an infinite but countable number of segments, namely $\mathbb{R} = \cup_{n=-\infty}^{+\infty}\delta_n$, with $\delta_n = [n\delta, (n+1)\delta[$ and arbitrary (but fixed) $\delta > 0$. We approximate the continuous random variable X_k by a discrete random variable $X_k^{(\delta)}$ with probability mass function $p_{X_k}^{(\delta)}$ given by

$$p_{X_k}^{(\delta)}(i) = \int_{i\delta}^{(i+1)\delta} f_{X_k}(x)\,dx\,, \quad i \in \mathbb{Z}\,.$$

Since δ can be chosen as small as one likes, the generalized continuous dice model can be approximated with arbitrary precision by a generalized discrete dice model, and, in particular, its probabilistic relation Q can be (elementwise) approximated by the dice-transitive probabilistic relation $Q^{(\delta)}$ of a generalized discrete dice model. Indeed, we have

$$q_{ij} = \int_{-\infty}^{+\infty} f_{X_i}(x) \left(\int_{-\infty}^{x} f_{X_j}(y)\, dy \right) dx$$

and

$$\begin{aligned} q_{ij}^{\delta} = & \textstyle\sum_k \sum_{l<k} \int_{k\delta}^{(k+1)\delta} f_{X_i}(x) \left(\int_{l\delta}^{(l+1)\delta} f_{X_j}(y)\, dy \right) dx \,+ \\ & \tfrac{1}{2} \textstyle\sum_k \int_{k\delta}^{(k+1)\delta} f_{X_i}(x) \left(\int_{k\delta}^{(k+1)\delta} f_{X_j}(y)\, dy \right) dx \,. \end{aligned}$$

Let now

$$\mu_\delta = \max_k \left(\int_{k\delta}^{(k+1)\delta} f_{X_i}(x)\, dx, \int_{k\delta}^{(k+1)\delta} f_{X_j}(y)\, dy \right) .$$

Then

$$\begin{aligned} q_{ij} - q_{ij}^{\delta} = & \textstyle\sum_k \int_{k\delta}^{(k+1)\delta} f_{X_i}(x) \left(\int_{k\delta}^{x} f_{X_j}(y)\, dy - \frac{1}{2} \int_{k\delta}^{(k+1)\delta} f_{X_j}(y)\, dy \right) dx \\ \le & \tfrac{1}{2} \textstyle\sum_k \int_{k\delta}^{(k+1)\delta} f_{X_i}(x)\, dx \int_{k\delta}^{(k+1)\delta} f_{X_j}(y)\, dy \\ \le & \tfrac{1}{2} \textstyle\sum_k \mu_\delta^2 \,. \end{aligned}$$

Completely analogous to the above, we obtain $q_{ji} - q_{ji}^{\delta} \le 1/2 \sum_k \mu_\delta^2$ and thus $|q_{ij} - q_{ij}^{\delta}| \le 1/2 \sum_k \mu_\delta^2$. When letting δ become arbitrarily small, we then obtain

$$\begin{aligned} \lim_{\delta\to 0} |q_{ij} - q_{ij}^{\delta}| \le & \lim_{\delta\to 0} \tfrac{1}{2} \textstyle\sum_k \mu_\delta^2 \\ = & \tfrac{1}{2} (\lim_{\delta\to 0} \mu_\delta)(\lim_{\delta\to 0} \textstyle\sum_k \mu_\delta) \\ \le & \lim_{\delta\to 0} \mu_\delta = 0 \,. \end{aligned}$$

The last step was obtained by considering that, when $\delta \to 0$, μ_δ is arbitrary small (since we are considering continuous r.v.), and it then holds that

$$\lim_{\delta\to 0} \left(\lim_{k\to+\infty} \sum_{l=-k}^{k} \mu_\delta \right) \le \lim_{k\to+\infty} \sum_{l=-k}^{k} \frac{1}{k} = \lim_{k\to+\infty} \frac{2k+1}{k} = 2 \,.$$

Finally, Theorem 4.2.1 implies that these probabilistic relations are indeed dice-transitive. ∎

To conclude this section, let us reformulate what we have obtained. The discrete dice model with independent random variables that are uniformly distributed on integer multisets is, as far as the transitivity of the generated probabilistic relation is concerned, a generic model, in the sense that all generalized dice models with independent r.v. generate dice-transitive probabilistic relations.

Of course, if the random variables of a generalized dice model possess distribution functions that obey certain constraints, then it is likely that the transitivity of the generated probabilistic relation is of a stronger type than dice-transitivity. In the remaining sections of this chapter, we will discuss certain of these constraints and their influence on the type of transitivity.

4.3 Dice with shifted distributions

As a first example of generalized dice models in which certain constraints are imposed on the distribution functions of the random variables, we consider the case where these r.v. possess cumulative distribution functions that are translated copies of a generic cumulative distribution function F_X. We will investigate the transitivity of the probabilistic relations generated by such restricted dice models and the notion of isostochastic transitivity, introduced in Section 2.4.4, will naturally come to the foreground.

PROPOSITION – 4.3.1: *Let the c.d.f. F_{X_i} of the independent random variables X_i, $i = 1, \ldots, m$, of a generalized dice model be arbitrary translations of the same c.d.f. F_X, i.e. $F_{X_i}(x) = F_X(x - t_i)$ for all i with arbitrary real t_i. If for all $u \neq v$ for which the equality*

$$\int_{-\infty}^{+\infty} F_X(x-u)\, dF_X(x) = \int_{-\infty}^{+\infty} F_X(x-v)\, dF_X(x) \tag{4.5}$$

holds, the integrals are either both 0 or both 1, then the probabilistic relation generated by the random variables is isostochastic transitive w.r.t. a function g that solely depends upon the generic c.d.f. F_X.

Proof:
We can assume without loss of generality that the indices of three random variables X_i, X_j, X_k are such that $q_{ij} \geq 1/2$ and $q_{jk} \geq 1/2$. The value of q_{ij} is computed as follows

$$q_{ij} = \int_{-\infty}^{+\infty} F_X(x - t_j)\, dF_X(x - t_i) = \int_{-\infty}^{+\infty} F_X(x + t_i - t_j)\, dF_X(x)\,.$$

Since F_X is nondecreasing and the last integral is equal to $1/2$ only when $t_i = t_j$, it is clear that $q_{ij} \geq 1/2$ implies $t_i \geq t_j$. Similarly, it holds that

$$q_{jk} = \int_{-\infty}^{+\infty} F_X(x + t_j - t_k)\, dF_X(x)\,,$$

with $t_j \geq t_k$. Finally,

$$q_{ik} = \int_{-\infty}^{+\infty} F_X(x + t_i - t_k)\, dF_X(x)\,,$$

and since $t_i - t_k = (t_i - t_j) + (t_j - t_k)$, we immediately obtain that

$$q_{ik} \geq \max(q_{ij}, q_{jk}) \geq \frac{1}{2}\,. \tag{4.6}$$

Let us first assume that $q_{ij} \neq 1$ and $q_{jk} \neq 1$. Then, due to the extra condition concerning (4.5), the differences $t_i - t_j$ and $t_j - t_k$ are unique, and so is their sum $t_i - t_k$. If $q_{ij} = 1$ or $q_{jk} = 1$, then, according to (4.6), also $q_{ik} = 1$. This proves that q_{ik} is a function of q_{ij} and q_{jk} on $[1/2,1]^2$, which we denote as $q_{ik} = g(q_{ij}, q_{jk})$ with g a $[1/2,1]^2 \to [1/2,1]$ function solely depending upon F_X. It is easy to verify that g is increasing and has $1/2$ as neutral element. For instance, if $q_{ij} = 1/2$ then condition (4.5) implies that $t_i = t_j$, whence $q_{ik} = g(1/2, q_{jk}) = q_{jk}$. Furthermore, g is symmetric and since $q_{ki} = 1 - q_{ik} \leq 1/2$, we can rewrite, using previously introduced notations, the functional relationship as $1 - \alpha_{ijk} = g(\beta_{ijk}, \gamma_{ijk})$ if $\beta_{ijk} \geq 1/2$, or equivalently

$$\alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 = \beta_{ijk} + \gamma_{ijk} - g(\beta_{ijk}, \gamma_{ijk}) \quad , \text{ if } \beta_{ijk} \geq 1/2 \,.$$

Since the above equality holds for all (i,j,k) for which $\beta_{ijk} \geq 1/2$, it follows that the probabilistic relation Q is cycle-transitive w.r.t. the self-dual function U_g^s defined in (2.51). Hence, according to the terminology introduced in Section 2.4.4, the probabilistic relation Q is g-isostochastic transitive. ∎

Note that $q_{ij} = 1$ implies that $F_X(x + t_i - t_j) = 1$ for all x for which $dF_X(x) \neq 0$. Hence, $q_{ij} = 1$ implies that $x + t_i - t_j \geq \tau_u$ should be satisfied for all $x \in [\tau_l, \tau_u]$, where τ_l and τ_u are the lower and upper bounds of the support of dF_X, or equivalently, $t_i - t_j \geq \tau_u - \tau_l = \tau$, where τ is the range of this support. This can therefore only occur if the distribution of X has finite support.

Finally, it must be emphasized that condition (4.5) is not only a sufficient but also a necessary condition for the g-isostochastic transitivity. However, in a continuous dice model, it is sufficient that the distribution of X has either infinite support or has as finite support a single interval. In a discrete dice model, it is sufficient that the probability mass function is strictly positive on a single interval of integers and zero elsewhere.

EXAMPLE – 4.3.2:
As a first example of a dice model with shifted distributions, let us consider the case of the exponential distribution with parameter λ, i.e. $F_X(x) = 1 - \exp(-\lambda x)$. Let us assume that the translational parameters for the three random variables X_i, X_j, X_k are such that $t_i \geq t_j \geq t_k$.
We compute

$$\begin{aligned} q_{ij} &= \text{Prob}\{X_i > X_j\} = \textstyle\int_{t_i}^{+\infty} \lambda e^{-\lambda(x-t_i)} [1 - e^{-\lambda(x-t_j)}]\, dx \\ &= 1 - \tfrac{1}{2} e^{-\lambda(t_i - t_j)} \,, \end{aligned}$$

from which it follows that $\exp(-\lambda(t_i - t_j)) = 2(1 - q_{ij})$. Similarly, it holds that $\exp(-\lambda(t_j - t_k)) = 2(1 - q_{jk})$. This leads to

$$q_{ik} = 1 - \frac{1}{2} e^{-\lambda(t_i - t_k)} = 1 - \frac{1}{2} e^{-\lambda(t_i - t_j)} e^{-\lambda(t_j - t_k)} = 1 - 2(1 - q_{ij})(1 - q_{jk}) \,.$$

Since $t_i \geq t_j \geq t_k$, it holds that $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$ and $q_{ki} \leq 1/2$, and the foregoing expression can be rewritten as

$$1 - \alpha_{ijk} = 1 - 2(1 - \beta_{ijk})(1 - \gamma_{ijk}) \,.$$

It then follows that Q is isostochastic transitive w.r.t. the function g defined by

$$g(x,y) = 1 - 2(1-x)(1-y)\,. \tag{4.7}$$

Using Proposition 2.4.10, we obtain the associated t-conorm S_g as

$$S_g(x,y) = x + y - xy\,,$$

which is the previously introduced probabilistic sum. ◀

EXAMPLE – 4.3.3:
As a second example, we consider the Gumbel distribution $G(\mu,\eta)$ as the generic distribution for a collection of shifted random variables. A continuous random variable X on $\mathbb{R}$ is said to be Gumbel-distributed with parameters μ and η, if it holds that:

$$f_X(x) = \mu e^{-\mu(x-\eta)} e^{-e^{-\mu(x-\eta)}}\,, \tag{4.8}$$

for any $x \in \mathbb{R}$. The corresponding c.d.f. is then given by

$$F_X(x) = e^{-e^{-\mu(x-\eta)}}\,.$$

The random variable X has expected value $\eta + C/\mu$ and variance $\pi^2/(6\mu^2)$, with C the Euler-Masceroni constant. It is known that if $X_1 \overset{d}{=} G(\mu,\eta_1)$ and $X_2 \overset{d}{=} G(\mu,\eta_2)$ are independent Gumbel-distributed random variables having the same variance (same μ), then the random variable $\max(X_1, X_2)$ is Gumbel-distributed with the same μ and with parameter $\eta = \ln(e^{\mu\eta_1} + e^{\mu\eta_2})/\mu$, whereas $X_1 - X_2$ is a random variable that has the logistic distribution, i.e.:

$$F_{X_1-X_2}(x) = \frac{1}{1 + e^{\mu(\eta_2-\eta_1-x)}}\,. \tag{4.9}$$

Let us assume that X_i, X_j, X_k are three random variables with distributions shifted by t_i, t_j, t_k from the generic Gumbel distribution $G(\mu,\eta)$. Then

$$q_{ij} = 1 - F_{X_i-X_j}(0) = \frac{e^{\mu(\eta_j-\eta_i)}}{1+e^{\mu(\eta_j-\eta_i)}} = \frac{e^{\mu\eta_j}}{e^{\mu\eta_i}+e^{\mu\eta_j}}\,.$$

Using the short notation $\lambda_i = \exp(\mu\,\eta_i)$, we obtain

$$q_{ij} = \frac{\lambda_i}{\lambda_i + \lambda_j}\,,$$

from which we immediately obtain that $q_{ij}/q_{ji} = \lambda_i/\lambda_j$. Since obviously equality (1.13) is satisfied for all (i,j,k), the probabilistic relation Q is multiplicatively transitive, or, equivalently, isostochastic transitive w.r.t. the function g defined in (2.54). ◀

4.4 Dice models with parametric random variables

4.4.1 Considered families

We now investigate independent continuous random variables with probability density functions taken from a one-parameter family of density functions. These families and density functions are listed in Table 4.1 (the variable parameter in all cases being λ, while the other parameters are treated as constants). In the case of normal distributions, for example, we only consider the one-parameter subfamily of normal distributions with varying expected value and constant variance.

Table 4.1: *Parametric families of continuous distributions.*

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, +\infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, +\infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R}$, $\mu > 0$	$x \in]-\infty, +\infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R}$, $a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$e^{-\lvert x-\lambda\rvert/\mu)}/(2\mu)$	$\lambda \in \mathbb{R}$, $\mu > 0$	$x \in]-\infty, +\infty[$
Normal	$e^{-(x-\lambda)^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}$, $\sigma > 0$	$x \in]-\infty, +\infty[$

4.4.2 Examples of multiplicative transitivity

4.4.2.1 Exponentially distributed dice

Let us consider the case of exponentially distributed dice, i.e. $X_i \stackrel{d}{=} E(\lambda_i)$. It then holds that

$$q_{ij} = \int_0^{+\infty} \lambda_i e^{-\lambda_i x} \left(\int_0^x \lambda_j e^{-\lambda_j y} dy \right) dx = \frac{\lambda_i}{\lambda_i + \lambda_j},$$

and it follows that $q_{ij}/q_{ji} = \lambda_i/\lambda_j$, which shows that Q is again multiplicatively transitive.

It is worthwhile to remark that the same transitivity property holds for the probabilistic relation Q generated by independent discrete random variables $X_i \stackrel{d}{=} G(p_i)$ that are geometrically distributed (i.e. $p_{X_i}(k) = p_i(1-p_i)^{k-1}, 0 < p_i < 1, k \geq 1$). Indeed, taking into consideration (4.3), we compute

$$\begin{aligned} q_{ij} = & \sum_{k=1}^{+\infty} (1-p_j)^{k-1} p_j \sum_{l=k+1}^{+\infty} (1-p_i)^{l-1} p_i + \\ & \frac{1}{2} \sum_{k=1}^{+\infty} (1-p_i)^{k-1} (1-p_j)^{k-1} p_i p_j = \frac{p_j(1-p_i/2)}{p_i + p_j - p_i p_j}, \end{aligned}$$

and one can easily verify that the equality $q_{ij}q_{jk}q_{ki} = (1-q_{ij})(1-q_{jk})(1-q_{ki})$ again holds. It is, after all, not so surprising that geometric distributions yield the same type of transitivity as exponential distributions, since the former can be regarded as a discretization of the latter.

4.4.2.2 Dice with a power-law distribution

The one-parameter power-law distributions mentioned in Table 4.1 form a subfamily of the family of Beta distributions as well as of the family of Pareto-distributions, the former ones having finite support, the latter ones having infinite support. We leave it to the reader to verify that in the case of Beta distributions we obtain $q_{ij} = \lambda_i/(\lambda_i+\lambda_j)$, while for the Pareto distributions we obtain $q_{ij} = \lambda_j/(\lambda_i+\lambda_j)$. In both cases we again obtain that the generated probabilistic relations are multiplicatively transitivity.

4.4.2.3 Gumbel-distributed dice

In Example 4.3.3, we have already introduced the two-parameter family of Gumbel distributions. By choosing $X_i \stackrel{d}{=} G(\mu, \lambda_i)$, the distribution of X_i can be regarded as the generic distribution $G(\mu, 0)$ shifted by λ_i. Hence, the result of Example 4.3.3 immediately applies, namely, the generated probabilistic relation Q is again multiplicatively transitive.

4.4.3 Other examples of isostochastic transitivity

Note that the remaining one-parameter families of distributions from Table 4.1 all concern distributions that for varying λ can be regarded as shifted versions of a single generic distribution. All these cases could therefore equally well have been treated before as examples of dice with shifted distributions, and moreover, we can already state, since the conditions of Proposition 4.3.1 are always fulfilled, that these families of distributions all generate a probabilistic relation that is g-isostochastic transitive, and hence also strongly stochastic transitive. It remains to characterize that function g for each of these families.

4.4.3.1 Dice with a unimodal uniform distribution

Let us consider independent random variables $X_i \stackrel{d}{=} U[\lambda_i, \lambda_i + a]$ uniformly distributed over the interval $[\lambda_i, \lambda_i + a]$ and let us furthermore assume without loss of generality that X_i, X_j, X_k are three such random variables for which it holds that $\lambda_i \geq \lambda_j \geq \lambda_k$. If $\lambda_i \geq \lambda_j + a$ then $q_{ij} = 1$ and if $\lambda_j \leq \lambda_i < \lambda_j + a$, then by straightforward computation we obtain

$$q_{ij} = 1 - \frac{(a + \lambda_j - \lambda_i)^2}{2a^2}.$$

Note that $\lambda_i \geq \lambda_j$ implies that $q_{ij} \geq 1/2$. Introducing the short notation $s_{ij} = \max(a + \lambda_j - \lambda_i, 0)$, it follows that if $\lambda_i \geq \lambda_j$ then $q_{ij} = 1 - s_{ij}^2/(2a^2)$. Similarly, since $\lambda_j \geq \lambda_k$, it holds that $q_{jk} = 1 - s_{jk}^2/(2a^2)$ and $q_{ik} = 1 - s_{ik}^2/(2a^2)$. Solving s_{ij} (resp. s_{jk}) in terms of q_{ij} (resp. q_{jk}), we find $s_{ij} = a(2(1-q_{ij}))^{1/2}$ (resp. $s_{jk} = a(2(1-q_{jk}))^{1/2}$). Since furthermore

$$s_{ik} = \max((a + \lambda_k - \lambda_j) + (a + \lambda_j - \lambda_i) - a, 0) = \max(s_{ij} + s_{jk} - a, 0)\,,$$

we obtain

$$q_{ik} = 1 - \frac{\left(\max(a\sqrt{2(1-q_{ij})} + a\sqrt{2(1-q_{jk})} - a, 0)\right)^2}{2a^2}\,,$$

which proves that the generated probabilistic relation Q is isostochastic transitive w.r.t. the function g defined by

$$g(x,y) = 1 - \frac{1}{2}\,\left(\max(\sqrt{2(1-x)} + \sqrt{2(1-y)} - 1, 0)\right)^2\,.$$

The associated t-conorm S_g, obtained using Proposition 2.4.10, is given by

$$S_g(x,y) = 1 - \left(\max(\sqrt{1-x} + \sqrt{1-y} - 1, 0)\right)^2\,,$$

which is known as the Schweizer-Sklar t-conorm $S_{1/2}^{\mathbf{SS}}$ with parameter value 1/2 [57].

4.4.3.2 Laplace-distributed dice (with constant variance)

Let $X_i \stackrel{d}{=} \mathrm{Lap}(\lambda_i, \mu_i)$ be Laplace-distributed random variables with parameters $\lambda_i, \mu_i > 0$, namely $f_{X_i}(x) = \exp(-|x-\lambda_i|/\mu_i)/(2\mu_i)$, then a straightforward computation leads to

$$q_{ij} = \begin{cases} 1 - \dfrac{1}{2(\mu_i^2 - \mu_j^2)}[\mu_i^2 e^{-(\lambda_i-\lambda_j)/\mu_i} - \mu_j^2 e^{-(\lambda_i-\lambda_j)/\mu_j}] & \text{, if } \lambda_i \geq \lambda_j\,, \\ \dfrac{1}{2(\mu_i^2 - \mu_j^2)}[\mu_i^2 e^{-(\lambda_j-\lambda_i)/\mu_i} - \mu_j^2 e^{-(\lambda_j-\lambda_i)/\mu_j}] & \text{, if } \lambda_i < \lambda_j\,, \end{cases}$$

which in the limit $\mu_i \to \mu$, $\mu_j \to \mu$, reduces to

$$q_{ij} = \begin{cases} 1 - \frac{1}{2}\left[1 + \frac{\lambda_i - \lambda_j}{2\mu}\right] e^{-(\lambda_i-\lambda_j)/\mu} & \text{, if } \lambda_i \geq \lambda_j\,, \\ \frac{1}{2}\left[1 + \frac{\lambda_j - \lambda_i}{2\mu}\right] e^{-(\lambda_j-\lambda_i)/\mu} & \text{, if } \lambda_i < \lambda_j\,. \end{cases}$$

Let f be the $[0,+\infty[\to]0,1/2]$ mapping defined by

$$f(x) = \frac{1}{2}\left(1 + \frac{x}{2}\right) e^{-x}\,,$$

then, if $\lambda_i \geq \lambda_j \geq \lambda_k$, we obtain

$$q_{ij} = 1 - f\left(\frac{\lambda_i - \lambda_j}{\mu}\right), q_{jk} = 1 - f\left(\frac{\lambda_j - \lambda_k}{\mu}\right), q_{ik} = 1 - f\left(\frac{\lambda_i - \lambda_k}{\mu}\right),$$

with $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$ and $q_{ik} \geq 1/2$. Since f is a one-to-one mapping, the generated probabilistic relation Q is isostochastic transitive w.r.t. the function g defined by

$$g(x,y) = 1 - f\left(f^{-1}(1-x) + f^{-1}(1-y)\right).$$

The associated strict t-conorm S_g is given by

$$S_g(x,y) = s^{-1}(s(x) + s(y)),$$

with additive generator

$$s(x) = f^{-1}\left(\frac{1-x}{2}\right).$$

4.4.3.3 Normally distributed dice (with same variance)

We use the notation $\Phi(x)$ for the c.d.f. of the standard normal distribution $\mathrm{N}(0,1)$ with expected value $\mu = 0$ and variance $\sigma^2 = 1$ (see Table 4.2). We will use the following well-known properties:

$$\Phi(-x) = 1 - \Phi(x), \quad \Phi^{-1}(x) = -\Phi^{-1}(1-x). \tag{4.10}$$

Let $X_i \stackrel{d}{=} \mathrm{N}(\mu_i, \sigma_i^2)$, $X_j \stackrel{d}{=} \mathrm{N}(\mu_j, \sigma_j^2)$ and $X_k \stackrel{d}{=} \mathrm{N}(\mu_k, \sigma_k^2)$, then, since $X_j - X_i \stackrel{d}{=} \mathrm{N}(\mu_j - \mu_i, \sigma_i^2 + \sigma_j^2)$, we obtain

$$q_{ij} = \mathrm{Prob}\{X_i > X_j\} = \mathrm{Prob}\{X_j - X_i < 0\} = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right).$$

Now let all X_i have the same variance σ^2, and let us without loss of generality assume that $\mu_i \geq \mu_j \geq \mu_k$, then

$$q_{ij} = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{2\sigma^2}}\right), \; q_{jk} = \Phi\left(\frac{\mu_j - \mu_k}{\sqrt{2\sigma^2}}\right), \; q_{ik} = \Phi\left(\frac{\mu_i - \mu_k}{\sqrt{2\sigma^2}}\right),$$

and $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$, $q_{ik} \geq 1/2$. Hence,

$$q_{ik} = \Phi\left(\Phi^{-1}(q_{ij}) + \Phi^{-1}(q_{jk})\right),$$

which proves that the probabilistic relation Q is g-isostochastic transitive with

$$g(x,y) = \Phi\left(\Phi^{-1}(x) + \Phi^{-1}(y)\right). \tag{4.11}$$

Note that due to (4.10) an alternative expression for the function g is

$$g(x,y) = 1 - \Phi\left(\Phi^{-1}(1-x) + \Phi^{-1}(1-y)\right) .$$

The associated strict t-conorm S_g is given by

$$S_g(x,y) = s^{-1}(s(x) + s(y))$$

with strict additive generator

$$s(x) = \Phi^{-1}\left(\frac{1-x}{2}\right) .$$

An overview of the results obtained in the present section is presented in Table 4.2 where for the random variables with parametric distributions defined in Table 4.1, we list the function g w.r.t. which the probabilistic relation Q is isostochastic transitive. In the cases of the unimodal uniform, Gumbel, Laplace

Table 4.2: *g-isostochastic transitivity for the dice models described in Table 4.1.*

Name	Function g
Exponential Beta Pareto Gumbel	$\dfrac{x\,y}{x\,y + (1-x)(1-y)}$ associated to t-conorm $S_2^{\mathbf{H}}$ (also valid for discrete geometric dice)
Uniform	$1 - \frac{1}{2}\left(\max(\sqrt{2(1-x)} + \sqrt{2(1-y)} - 1, 0)\right)^2$ associated to t-conorm $S_{1/2}^{\mathbf{SS}}$
Laplace	$1 - f\left(f^{-1}(1-x) + f^{-1}(1-y)\right)$ with $f(x) = \frac{1}{2}\left(1 + \frac{x}{2}\right) e^{-x}$
Normal	$\Phi\left(\Phi^{-1}(x) + \Phi^{-1}(y)\right)$ with $\Phi(x) = (\sqrt{2\pi})^{-1} \int_{-\infty}^{x} e^{-t^2/2} dt$

and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions for which, as we will see in the next section, a lot of simplifying steps in the computations allow to maintain the two parameters as free parameters.

4.5 Normally distributed dice

Let us again consider a collection of normally distributed random variables $X_i \stackrel{d}{=} \mathrm{N}(\mu_i, \sigma_i^2)$. We know from the previous section that

$$q_{ij} = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right),\ q_{jk} = \Phi\left(\frac{\mu_j - \mu_k}{\sqrt{\sigma_j^2 + \sigma_k^2}}\right),\ q_{ik} = \Phi\left(\frac{\mu_i - \mu_k}{\sqrt{\sigma_i^2 + \sigma_k^2}}\right).$$

Introducing the notation $\phi_{ij} = \sqrt{\sigma_i{}^2 + \sigma_j{}^2}$, it follows from $\mu_i - \mu_k = (\mu_i - \mu_j) + (\mu_j - \mu_k)$, that

$$\phi_{ik}\Phi^{-1}(q_{ik}) = \phi_{ij}\Phi^{-1}(q_{ij}) + \phi_{jk}\Phi^{-1}(q_{jk})\,, \tag{4.12}$$

an equality which, since $\phi_{ik} = \phi_{ki}$, can be rewritten as

$$\phi_{ij}\Phi^{-1}(q_{ij}) + \phi_{jk}\Phi^{-1}(q_{jk}) + \phi_{ki}\Phi^{-1}(q_{ki}) = 0\,.$$

This formula turns out to be a key element in the proof of the following proposition.

PROPOSITION – 4.5.1: *The probabilistic relation generated by a collection of independent normal random variables is moderately stochastic transitive.*

Proof:
Let us consider the case $q_{ij} \geq 1/2$ and $q_{jk} \geq 1/2$. It then follows that $\mu_i \geq \mu_j \geq \mu_k$, with as a consequence that also $q_{ik} \geq 1/2$. This means that $\gamma_{ijk} \geq \beta_{ijk} \geq 1/2$ and $\alpha_{ijk} = q_{ki}$. We have to prove that $1 - \alpha_{ijk} = q_{ik} \geq \min(\beta_{ijk}, \gamma_{ijk}) = \min(q_{ij}, q_{jk})$. Since Φ^{-1} is a strictly increasing function, this is equivalent to proving that the inequality $\Phi^{-1}(q_{ik}) \geq \min(\Phi^{-1}(q_{ij}), \Phi^{-1}(q_{jk}))$ is fulfilled. Using (4.12), we obtain that

$$\begin{aligned}\Phi^{-1}(q_{ik}) &= \frac{\phi_{ij}}{\phi_{ik}}\Phi^{-1}(q_{ij}) + \frac{\phi_{jk}}{\phi_{ik}}\Phi^{-1}(q_{jk}) \\ &\geq \frac{\phi_{ij} + \phi_{jk}}{\phi_{ik}} \min\left(\Phi^{-1}(q_{ij}), \Phi^{-1}(q_{jk})\right)\,.\end{aligned}$$

From the definition of ϕ_{ij}, it follows that $\phi_{ij} > 0$ and furthermore it can easily be shown that $|\phi_{jk}^2 - \phi_{ij}^2| \leq \phi_{ik}^2 \leq \phi_{ij}^2 + \phi_{jk}^2$, which implies that the numbers ϕ_{ik}^2, ϕ_{ij}^2, and ϕ_{jk}^2 are triangular numbers, since they satisfy the classical triangular conditions. From the rightmost inequality of this double inequality, we derive that

$$\phi_{ij} + \phi_{jk} = \sqrt{\phi_{ij}^2 + \phi_{jk}^2 + 2\phi_{ij}\phi_{jk}} \geq \sqrt{\phi_{ij}^2 + \phi_{jk}^2} \geq \phi_{ik}\,,$$

which completes the proof. ∎

Interestingly, the reverse statement of the above proposition also holds.

PROPOSITION – 4.5.2: *Moderate stochastic transitivity is the characteristic transitivity for 3-dimensional dice models with independent normal random variables.*

Proof:
Let us consider the case $q_{jk} \geq q_{ij} \geq 1/2$, the case $q_{ij} \geq q_{jk} \geq 1/2$ is completely analogous. In the proof of Proposition 4.5.1 we obtained

$$\Phi^{-1}(q_{ik}) = \frac{\sqrt{\sigma_i^2+\sigma_j^2}}{\sqrt{\sigma_i^2+\sigma_k^2}}\Phi^{-1}(q_{ij}) + \frac{\sqrt{\sigma_j^2+\sigma_k^2}}{\sqrt{\sigma_i^2+\sigma_k^2}}\Phi^{-1}(q_{jk})\,. \tag{4.13}$$

When considering (4.13) as a function of σ_i over $[0,+\infty]$ with all other values being treated as real strictly positive constants and $\sigma_k < \sigma_j$, we see that it is strictly decreasing. Letting σ_i vary from $+\infty$ to 0, $\Phi^{-1}(q_{ik})$ varies from $\Phi^{-1}(q_{ij})$ to

$$\frac{\sigma_j}{\sigma_k}\Phi^{-1}(q_{ij}) + \frac{\sqrt{\sigma_j^2+\sigma_k^2}}{\sigma_k}\Phi^{-1}(q_{jk})\,.$$

Treating the above quantity as a function of σ_j over $]\sigma_k,+\infty]$, we see that it is strictly increasing and goes to infinity when $\sigma_j = +\infty$.

Concluding, we can let $\Phi^{-1}(q_{ik})$ vary from $\Phi^{-1}(q_{ij})$ to $+\infty$. In other words, we can let q_{ik} vary from $\min(q_{ij}, q_{jk})$ to 1. As Proposition 4.5.1 already proved that all dice models with independent normal random variables are at least moderately stochastic transitive, this concludes the proof. ∎

4.6 The uniform distribution revisited

In this section we consider a collection of independent uniformly distributed random variables $X_i \stackrel{d}{=} U[\lambda_i, \lambda_i + a_i]$, satisfying different constraints than those from Subsection 4.4.3.1. The obtained characteristic form of cycle-transitivity will turn out not to be a type of isostochastic transitivity but instead it will be g-stochastic transitivity with the function g given by $g(x,y) = 1-2(1-x)(1-y)$. Note that g-isostochastic transitivity with the same function g was already encountered in Example 4.3.2. For ease of notation, we introduce the values $\mu_i = \lambda_i + a_i$. We now impose on the collection of random variables the constraint that, taking any 3 r.v. X_i, X_j, X_k from the collection, the indices of these r.v. can be renumbered such that $\lambda_k \leq \lambda_j \leq \lambda_i \leq \mu_k \leq \mu_j \leq \mu_i$. To set the mind, Figure 4.1 shows the p.d.f. of 3 r.v. satisfying the imposed constraints. We obtain

$$q_{ij} = 1 - \frac{(\mu_j-\lambda_i)^2}{2a_ia_j}\,,\; q_{jk} = 1 - \frac{(\mu_k-\lambda_j)^2}{2a_ja_k}\,,\; q_{ki} = \frac{(\mu_k-\lambda_i)^2}{2a_ia_k}\,.$$

It is obvious that for the given equalities it holds that $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$ and $q_{ki} \leq 1/2$, implying that $\alpha_{ijk} = q_{ki}$. We now determine all possible values for

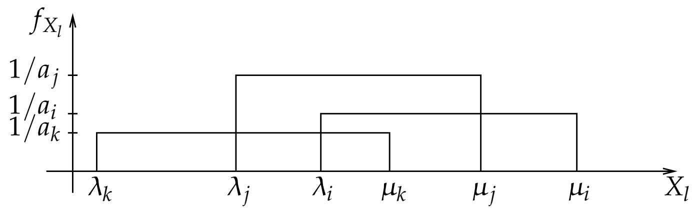


Figure 4.1: *Dice model satisfying the constraints.*

q_{ki}, under the constraints mentioned above, while keeping q_{ij} and q_{jk} fixed. To that extent, we will fix $\lambda_i, \mu_i, \lambda_j, \mu_j$ and will let μ_k vary between λ_i and μ_j, taking the appropriate value for λ_k such that q_{jk} is invariant.

Consider a new set of r.v. X_i, X_j, $X_k^{(x)}$ with $X_k^{(x)} \overset{d}{=} U[\lambda_k^{(x)}, \mu_k^{(x)}]$ and $\mu_k^{(x)} - \lambda_k^{(x)} = a_k^{(x)}$, $\mu_k^{(x)} = \mu_k + x$, $x \in [\lambda_i - \mu_k, \mu_j - \mu_k]$, and corresponding values $\lambda_k^{(x)}$ and $a_k^{(x)}$ such that $q_{jk}^{(x)} = Q(X_j, X_k^{(x)}) = q_{jk}$. Note that, since $q_{jk} \geq 1/2$ we are assured that the obtained values for $\lambda_k^{(x)}$ will always be less or equal to λ_j. The equality $Q(X_j, X_k^{(x)}) = q_{jk}$ is equivalent to

$$\frac{(\mu_k^{(x)} - \lambda_j)^2}{2a_j a_k^{(x)}} = \frac{(\mu_k - \lambda_j)^2}{2a_j a_k}\,,$$

which in turn is equivalent to

$$a_k^{(x)} = \frac{a_k(\mu_k - \lambda_j + x)^2}{(\mu_k - \lambda_j)^2}\,.$$

We then obtain that

$$q_{ki}^{(x)} = \frac{(\mu_k^{(x)} - \lambda_i)^2}{2a_i a_k^{(x)}} = \frac{(\mu_k - \lambda_i + x)^2}{2a_i a_k}\frac{(\mu_k - \lambda_j)^2}{(\mu_k - \lambda_j + x)^2}\,.$$

Considering $q_{ki}^{(x)}$ to be a continuous function of x, with $x \in [\lambda_i - \mu_k, \mu_j - \mu_k]$, we can take its derivative in order to obtain the minimum and maximum value for $q_{ki}^{(x)}$.

$$D(q_{ki}^{(x)}) = \frac{(\lambda_i - \lambda_j)(\mu_k - \lambda_j)^2}{a_i a_k}\frac{\mu_k - \lambda_i + x}{(\mu_k - \lambda_j + x)^3}\,.$$

Thus, $q_{ki}^{(x)}$ is increasing over $[\lambda_i - \mu_k, \mu_j - \mu_k]$. The minimal value for q_{ki} is therefore obtained by setting $x = \lambda_i - \mu_k$ and is given by

$$q_{ki}^{(\lambda_i - \mu_k)} = 0\,.$$

The maximal value for q_{ki}, under the mentioned constraints, is then obtained by setting $x = \mu_j - \mu_k$ and is given by

$$q_{ki}^{(\mu_j - \mu_k)} = \frac{(\mu_k - \lambda_j)^2(\mu_j - \lambda_i)^2}{2a_i a_k a_j^2}\,. \tag{4.14}$$

The upper bound (4.14) for q_{ki} under the mentioned constraints holds for any choice of $\lambda_i, \mu_i, \lambda_j, \mu_j$. The right-hand side of the above equation is therefore the maximal value for $\alpha_{ijk} = q_{ki}$ (in function of $\lambda_i, \mu_i, \lambda_j, \mu_j$) and it can be rewritten as

$$q_{ki} \leq 2(1 - q_{ij})(1 - q_{jk}),$$

with $\min(q_{ij}, q_{jk}) \geq 1/2$. The models investigated in this section are therefore cycle-transitive w.r.t. the upper bound

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma) & , \beta \geq 1/2, \\ 2 & , \beta < 1/2, \end{cases}$$

with $g(x, y) = 1 - 2(1 - x)(1 - y)$. Note that the above upper bound can be recast in another nice equivalent form, namely

$$U'_g(\alpha, \beta, \gamma) = \begin{cases} \beta\gamma + (1 - \beta)(1 - \gamma) & , \beta \geq 1/2, \\ 2 & , \beta < 1/2. \end{cases}$$

The true logic of this world is in the calculus of probabilities.

— JAMES C. MAXWELL

5
Dice Games

In the previous chapter, a method was developed for comparing random variables by pairwisely coupling them with a certain copula and defining a probabilistic relation. In this chapter, we show that the same method can be used to describe certain games. We focus here on independent random variables, i.e. r.v. coupled by the $T_{\mathbf{P}}$-copula, some games with differently coupled r.v. will be discussed in Chapter 7. In the present chapter, a class of symmetric games will be proposed and the necessary and sufficient conditions for such a game to possess an optimal strategy will be laid bare along with the corresponding optimal strategies. These results can be found in [26, 27, 30].

5.1 Introduction

Two mathematicians want to play a game and they happen to have at their disposal two fair dice with n faces. As they love natural numbers (but for some reason despise zero), they each take a dice and write on each face of the dice such a non-negative number. They then throw their dice and the dice with the highest number showing up on the bottom face wins the game, if the two numbers happen to be equal they play a draw. The players are of course not stupid — they are mathematicians after all — so in their first game they end up writing huge numbers on the faces. They instantaneously realize some limitation has to be imposed on the numbers to be written on the faces and arrive at the natural limitation that the sum of the numbers on the faces of a dice must equal some fixed number σ. After some discussion they coin the name of their game as an (n, σ) game.

Since they each can choose their numbers independently from the other, both players have in principle the same odds to win. However, not all combinations of numbers on the faces are good choices. There might be special combinations that should be preferred to others: some strategies might be optimal while others are suboptimal. At first glance they cannot deduce which strategies should be preferred, so each mathematician retreats to her study for further investigation. If they do their job right, the results of their quest will agree with the results that are obtained in the following sections. We now continue with a more formal description of the game.

5.2 Description of the game

The (n, σ) dice game is a game played between two players who want to obtain the highest individual profit. Both players choose independently a dice from the collection of (n, σ) dice. An (n, σ) dice is a fair, not necessarily materializable dice with n faces, each face containing a strictly positive number of eyes and the sum of the eyes on all faces being equal to the given σ. One can therefore also represent an (n, σ) dice by an (n, σ) partition, introduced in Definition 1.4.2.

Once the players have chosen their own dice (note that they might have selected the same kind), the (n, σ) dice game is played in one or more rounds. At the beginning of each round, both players place a bet of, say, €1, then independently roll their dice and compare the number of eyes on the bottom face: the dice that falls on the face with the highest number of eyes wins the round. The winner takes all and a new round can start. If the faces show the same number of eyes, the round ends in a draw: there is no winner, both players get their €1 back before starting a new round. Since the two players have as objective to win the game, and since each round of the game proceeds under the same conditions (same strategy, same bet), they want to choose from the collection a dice that maximizes their winning probability.

In order to compute the winning probability, let us regard dice A_i of player 1 and dice A_j of player 2 as independent discrete random variables, uniformly distributed on the multiset consisting of the number of eyes, for each face, of dice A_i and A_j, respectively. A uniform distribution on a finite multiset is, in general, equivalent to a discrete distribution on an ordinary set endowed with a rational probability mass function. The winning probability q_{ij} of dice A_i w.r.t. dice A_j is the probability that, after the dice are thrown, the number on the bottom face of A_i is strictly greater than the number on the bottom face of A_j plus one half of the probability that both numbers are equal, thus

$$q_{ij} = \mathrm{Prob}\{A_i > A_j\} + \frac{1}{2}\,\mathrm{Prob}\{A_i = A_j\}\,. \tag{5.1}$$

This probabilistic relation is therefore the same as the one introduced in Chapter 3. Note that if both players have chosen the same dice, say A_i, then as they roll it independently, they obviously have the same winning probability $q_{ii} = 1/2$.

On the complete collection $\{A_i\}_{i=1}^k$ of (n, σ) dice, k denoting the number of partitions of σ into n parts, we consider the probabilistic relation $Q = [q_{ij}]$ consisting of the winning probabilities between all couples of dice, or equivalently, all couples of (n, σ) partitions. This set of dice therefore encompasses a dice model that generates the probabilistic relation Q.

Although the games considered are played with dice of the same type (same n and same σ), we will need to compare as well dice with same n but different σ. We therefore generalize (5.1) by allowing the sum of the integers on dice A_i and A_j to differ.

In the next section, we characterize the dice game described above in the formal setting of game theory. Section 5.4 then gives in the form of a theorem and a number of propositions and corollaries a clear answer to the following questions: for which values of n and σ do there exist optimal strategies, and if such strategies exist, how many are they and what is their precise form? Section 5.5 contains the proof of the results covered in Section 5.4. To make these proofs as comprehensible as possible, examples of (n, σ) dice games and their strategies will be used at different places to illustrate the theoretical results. A table with the number of optimal strategies for some values of n and σ is given in Table 7.3 on page 134.

5.3 Game-theoretic characterization of the (n, σ) dice game

For a short explanation of the used game-theoretical concepts, we refer to Section 1.3. The (n, σ) dice A_i, with $i = 1, 2, \ldots, k$, are the pure strategies. Let us denote the set of all pure strategies as A. The problem of finding the best dice therefore amounts to finding the optimal strategies of the game. In this respect, we define the payoff function $p^{(1)} : A \times A \to [-1/2, 1/2]$ of player 1 by

$$p^{(1)}(A_i, A_j) = p_{ij}^{(1)} = q_{ij} - \frac{1}{2}, \tag{5.2}$$

where the first argument A_i denotes the strategy of the first player and the second argument A_j the strategy of the second player. It follows that the payoff function $p^{(2)}$ of player 2 is then given by

$$p^{(2)}(A_i, A_j) = p_{ij}^{(2)} = (1 - q_{ij}) - \frac{1}{2} = -p_{ij}^{(1)}, \tag{5.3}$$

where the meaning of the two arguments is the same as in (5.2). As short notation and in accordance with Section 1.3, we use $a_{ij}^1 = p^{(1)}(A_i, A_j)$ and $a_{ij}^2 = p^{(2)}(A_i, A_j)$. Note that the payoff $2a_{ij}^d$ lies in the interval $[-1, 1]$ and is for $d \in \{1, 2\}$ nothing else than the expected gain (expressed in €) of player d in a single round (when both players bet € 1). As follows from (5.3), the game is a symmetric matrix game. The example payoff matrix given in Figure 1.2 on page 13 corresponds to the $(6, 12)$ dice game.

The rest of this chapter is devoted to the characterization of all optimal strategies of the (n, σ) dice games. It must be noted, however, that not all (n, σ) dice games have optimal strategies and for these games we obviously cannot state the strategies a player should pick to maximize her winning probabilities. We will use the notation π_i from partition theory instead of the notation A_i which was used in previous chapters.

An (n, σ) partition π_i is an optimal strategy in the (n, σ) dice game if it does not lose from any other strategy. Therefore, π_i is optimal if and only if $a_{ij}^1 \geq 0$ or, equivalently, $q_{ij} = Q(\pi_i, \pi_j) \geq 1/2$, for all (n, σ) partitions π_j. The process of proving that an (n, σ) partition is optimal or not will thus involve checking whether $q_{ij} = Q(\pi_i, \pi_j) \geq 1/2$ for all (n, σ) partitions π_j or not. When it is better suited to explicitly mention the (n, σ) partitions defining q_{ij}, we will use the notation Q_{π_i, π_j} instead of $Q(\pi_i, \pi_j)$. The same technique and notations will be used in Chapter 7 when determining the optimal strategies of the games that are considered there.

5.4 Optimal strategies for (n, σ) dice games

In this section, we merely state the main results characterizing the optimal strategies for (n, σ) dice games. In this and upcoming chapters, we make use of the multiplicity representation of a partition, introduced in Definition 1.4.3.

A first important observation is that not all (n, σ) dice games possess one or more optimal strategies. The following theorem formulates the necessary and sufficient conditions for an (n, σ) dice game to have at least one optimal strategy.

THEOREM – 5.4.1: *An (n, σ) dice game has at least one optimal strategy if and only if one of the following six mutually exclusive conditions is satisfied:*

(i) $n \leq 2$

(ii) $(n, \sigma) = (3, 7)$

(iii) $(n, \sigma) = (3, 8)$

(iv) $(n, \sigma) = (2l, 4l + 1)\,, l > 1$

(v) $n > 2$ *and there exist* $a, b, k \in \mathbb{N}$ *such that*

$$\begin{cases} n = (a + b)\,k - b \\ \sigma = n\,k \end{cases} \tag{5.4}$$

(vi) $n > 2$ *and there exist* $a, b, k \in \mathbb{N}$ *such that*

$$\begin{cases} n = (a + b)\,k \\ \sigma = (n + b)\,k \\ a \neq 0 \wedge b \neq 0 \end{cases} \tag{5.5}$$

While the above theorem characterizes all (n, σ) dice games that possess optimal strategies, the propositions below state the number of optimal strategies and their explicit form. We start by handling the special cases.

PROPOSITION – 5.4.2:

1. *The $(1, \sigma)$ dice game: the unique strategy (σ^1) is optimal.*
2. *The $(2, \sigma)$ dice game: all $\lfloor \frac{\sigma}{2} \rfloor$ strategies $(k^1(\sigma - k)^1)$, $0 < k \leq \lfloor \frac{\sigma}{2} \rfloor$, are optimal.*
3. *The $(3, 7)$ dice game: $(1^1 3^2)$ is the only optimal strategy.*
4. *The $(3, 8)$ dice game: $(1^1 3^1 4^1)$ is the only optimal strategy.*
5. *The (n, n) dice game: the unique strategy (1^n) is optimal.*
6. *The $(2n, 4n + 1)$ dice game, $n > 1$: $(1^{n-1} 2^1 3^n)$ is the only optimal strategy.*

The next proposition discusses the dice games of type (5.4), excluding the (n, n) dice game as it has already been considered in the above proposition.

PROPOSITION – 5.4.3: *All (n, σ) dice games, with $n \neq \sigma$, satisfying* (5.4) *have exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ optimal strategies and their multiplicity representation is given by $(1^a 2^b 3^a 4^b \ldots (2k-2)^b (2k-1)^a)$, where a, b are different but k is the same for each optimal strategy.*

EXAMPLE – 5.4.4:
The $(6, 12)$ dice game, for which the payoff matrix is given in Figure 1.2, has the following pure strategies.

$$\begin{array}{lll} \pi_1 = (1,1,1,1,1,7) & \pi_5 = (1,1,1,2,2,5) & \\ \pi_2 = (1,1,1,1,2,6) & \pi_6 = (1,1,1,2,3,4) & \pi_9 \;= (1,1,2,2,3,3) \\ \pi_3 = (1,1,1,1,3,5) & \pi_7 = (1,1,1,3,3,3) & \pi_{10} = (1,2,2,2,2,3) \\ \pi_4 = (1,1,1,1,4,4) & \pi_8 = (1,1,2,2,2,4) & \pi_{11} = (2,2,2,2,2,2) \end{array}$$

With $n = 6$ and $\sigma = 12$, system (5.4) has the following solutions ($k = 2$): $a = 3$ and $b = 0$, $a = 2$ and $b = 2$, $a = 1$ and $b = 4$, $a = 0$ and $b = 6$. According to Proposition 5.4.3, the multiplicity representations of the corresponding optimal strategies are given by

$$\begin{array}{ll} a = 3 \text{ and } b = 0 : (1^3 3^3)\,, & a = 1 \text{ and } b = 4 : (1^1 2^4 3^1)\,, \\ a = 2 \text{ and } b = 2 : (1^2 2^2 3^2)\,, & a = 0 \text{ and } b = 6 : (2^6)\,. \end{array}$$

These clearly correspond to the partitions π_7, π_9, π_{10} and π_{11}. One can verify that for any of the above a and b it indeed holds that $\lfloor \frac{a}{k-1} \rfloor + \lfloor \frac{b}{k} \rfloor + 1 = 4$. ◀

Finally, the games of type (5.5) are considered.

PROPOSITION – 5.4.5: *All (n, σ) dice games satisfying* (5.5) *have exactly one optimal strategy $(1^a 2^b 3^a 4^b \ldots (2k-1)^a (2k)^b)$.*

EXAMPLE – 5.4.6:

(i) The $(6, 21)$ game has 110 strategies and satisfies (5.5) with $a = b = 1$ and $k = 3$. The unique optimal strategy for this game is given by $(1, 2, 3, 4, 5, 6)$, the most common of all dice.

(ii) The $(8, 22)$ dice game has 116 strategies and satisfies (5.5), with $k = 2$, $a = 1$ and $b = 3$. The unique optimal strategy for this game is given by $(1^1 2^3 3^1 4^3)$. ◀

The above propositions imply the following corollaries, which are statements about certain types of diophantine systems. The first corollary is implied by Proposition 5.4.3.

COROLLARY – 5.4.7: *For given values of n and σ, $n \neq \sigma$, the entity $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor$ is an invariant of the solution space of system* (5.4)*. If this system has a solution, then it has exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ solutions.*

On the other hand, Proposition 5.4.5 implies the following proposition.

COROLLARY – 5.4.8: *For given values of n and σ, system* (5.5) *has at most one solution.*

5.5 Proof of the main results

We will use the concepts of decremented and incremented partitions, which we first introduce here.

DEFINITION – 5.5.1:

1. *The decremented partition* $\delta(\pi, m)$ *corresponding to a given* (n, σ) *partition* $\pi = (i_1, i_2, \ldots, i_n)$ *is the* $(n, \sigma - 1)$ *partition obtained by decrementing the element* i_m *of* π, $1 \leq m \leq n$, *where it is assumed that* $i_m \neq 1$.

2. *The incremented partition* $\nu(\pi, m)$ *corresponding to a given* (n, σ) *partition* $\pi = (i_1, i_2, \ldots, i_n)$ *is the* $(n, \sigma + 1)$ *partition obtained by incrementing the element* i_m *of* π, $1 \leq m \leq n$.

The proof of the theorem and propositions from Section 5.4 is realized by dividing the collection of (n, σ) partitions in different classes and by determining for each class separately the partitions that are optimal, if there are any. With each class of partitions we will, when necessary, consider one or more subcases.

Cases 1–5 cover the (n, σ) dice games specified in items (i)–(iii) from Theorem 5.4.1 and items 1–5 from Proposition 5.4.2. After considering these cases, we will introduce the increment/decrement operation, which will play a central role when considering the subsequent cases. Case 6 considers a specific type of strategies for which it is proven that they cannot be optimal. Case 7 considers another class of strategies, for which it is shown that there exists a limited subset of optimal strategies. In this subset, the only strategies not yet covered in previous cases, are those stated in item 6 of Proposition 5.4.2. The proof of item (iv) in Theorem 5.4.1 is then immediate. Finally, Case 8 considers all remaining strategies, not yet covered by the previous cases. These strategies can be nicely characterized and we will divide them in three subclasses, each leading to a subcase in the proof. Subcases 8.1 and 8.2 are concerned with subsets of strategies that will be proven not to be optimal, while in subcase 8.3 we consider the remaining strategies, which are shown to be the optimal strategies mentioned in Propositions 5.4.3 and 5.4.5. First, we will prove that these strategies are indeed optimal, then we will prove that they only exist when either condition (5.4) or condition (5.5) is satisfied. Also, the number of these optimal strategies will be counted in order to finalize the proof of Propositions 5.4.3 and 5.4.5. Corollaries 5.4.7 and 5.4.8 will immediately follow from these results.

5.5.1 Proofs for some special (n, σ) dice games

As mentioned above, we first consider some special cases that cannot be handled in a more general way. These all have optimal strategies.
Case 1: The $(1, \sigma)$ dice game.
There is only one strategy in this type of game, namely (σ^1), and this strategy is therefore optimal. ∎

Case 2: The $(2, \sigma)$ dice game.
For any two $(2, \sigma)$ partitions π_1 and π_2, it holds that $Q_{\pi_1,\pi_2} = 1/2$. Indeed, for any two distinct $(2, \sigma)$ partitions $\pi_1 = (a_1, a_2)$ and $\pi_2 = (b_1, b_2)$, we have that either $a_1 < b_1 \leq b_2 < a_2$ or $b_1 < a_1 \leq a_2 < b_2$, from which it follows that $Q_{\pi_1,\pi_2} = 1/2$. Therefore, any $(2, \sigma)$ partition is an optimal strategy. Moreover, it is obvious that there exist exactly $\lfloor \sigma/2 \rfloor$ such $(2, \sigma)$ partitions. ∎
Case 3: The $(3, 7)$ dice game.
In the $(3, 7)$ dice game, it holds that $(1, 3, 3)$ is the only optimal strategy. Indeed, there are four $(3, 7)$ partitions: $(1, 1, 5)$, $(1, 2, 4)$, $(1, 3, 3)$, $(2, 2, 3)$. Easy calculations support the stated result. ∎
Case 4: The $(3, 8)$ dice game.
In the $(3, 8)$ dice game, it holds that $(1, 3, 4)$ is the only optimal strategy. Indeed, there are five $(3, 8)$ partitions: $(1, 1, 6)$, $(1, 2, 5)$, $(1, 3, 4)$, $(2, 2, 4)$, $(2, 3, 3)$. Again, easy calculations support the stated result. ∎
Case 5: The (n, n) dice game.
There is only one strategy in this dice game, namely (1^n), which is therefore optimal. ∎

5.5.2 Decremented and incremented partitions reconsidered

In what follows, we can exclude the above special cases. Before going further, we need to introduce some concepts related to decremented partitions.

DEFINITION – 5.5.2: *Consider an (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$. Let m ($1 \leq m \leq n$) be such that $i_m \neq 1$ and*

$$(\forall\, 1 \leq j \leq n)\, (i_j \neq 1 \Rightarrow Q_{\pi,\delta(\pi,j)} \leq Q_{\pi,\delta(\pi,m)})\,. \tag{5.6}$$

Such values of m will be called max-decrement positions.

Property (5.6) specifies that the probability that the original partition π wins from the decremented partition $\delta(\pi, i)$ is highest when i is a max-decrement position. Note that such a value is not necessarily unique. For example, the $(10, 39)$ partition $\pi = (1, 1, 1, 3, 3, 4, 6, 6, 6, 8)$ has 6, 7, 8 or 9 as possible max-decrement positions, where $\delta(\pi, 6) = (1, 1, 1, 3, 3, 3, 6, 6, 6, 8)$, $\delta(\pi, 7) = \delta(\pi, 8) = \delta(\pi_1, 9) = (1, 1, 1, 3, 3, 4, 5, 6, 6, 8)$ and $Q_{\pi,\delta(\pi,6)} = Q_{\pi,\delta(\pi,7)} = 1/2 + 3/200$.

Since (n, n) dice games are excluded, at least one such max-decrement position exists. Clearly, for any $i_k \neq 1$ it holds that

$$Q_{\pi,\delta(\pi,k)} = Q_{\pi,\pi} + \frac{t_{i_k} + t_{i_k-1}}{2n^2} = \frac{1}{2} + \frac{t_{i_k} + t_{i_k-1}}{2n^2}\,.$$

Also, incrementing i_k in π gives rise to the following equality:

$$Q_{\pi,\nu(\pi,k)} = \frac{1}{2} - \frac{t_{i_k} + t_{i_k+1}}{2n^2}\,.$$

Next, suppose we increment i_k and decrement i_l, $i_l \neq 1$ and $l \neq k$, in an (n, σ) partition π_1, then we obtain an (n, σ) partition π_2. We call this operation an increment/decrement operation. The following equality then holds.

$$Q_{\pi_1,\pi_2} = \frac{1}{2} + \frac{t_{i_l} + t_{i_l-1} - t_{i_k} - t_{i_k+1}}{2n^2} .$$

In the next cases, when proving that a given partition π_1 is not an optimal strategy, we will construct a partition π_2 such that $Q_{\pi_1,\pi_2} < 1/2$ by means of increment/decrement operations. Obviously, this construction of a partition π_2 that wins from partition π_1 is in general not unique.

5.5.3 Towards a special case: the $(2l, 4l+1)$ dice games

We now divide the set of all remaining (n, σ) partitions into three classes and investigate each class separately.
Case 6: Consider an (n, σ) partition $\pi_1 = (1^{t_1} 2^{t_2} \ldots)$ such that

$$(\exists\, j > 0)\; (t_j = 0 \wedge t_{j+1} = 0 \wedge t_{j+2} \neq 0) . \tag{5.7}$$

Partitions satisfying (5.7) are not optimal. Indeed, take such a partition π_1 and decrement an occurrence of $j+2$ by 2 and increment two different elements l (if $t_{j+2} > 1$ then choose l to be another occurrence of $j+2$) and m from partition π_1. The resulting partition π_2 wins from π_1. Indeed, it holds that

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_l + t_{l+1} + t_m + t_{m+1} - t_{j+2}}{2n^2} .$$

From the right-hand side it is seen that Q_{π_1,π_2} is strictly smaller than $1/2$, since it clearly holds that $t_l > 0$ and $t_m > 0$ and that $l = j+2$ when $t_{j+2} > 1$. ■

EXAMPLE – 5.5.3:

(i) Consider the $(4, 15)$ partition $\pi_1 = (1, 4, 4, 6)$ for which formula (5.7) is satisfied for $j = 2$. For the $(4, 15)$ partition $\pi_2 = (2, 2, 5, 6)$ it holds that $Q_{\pi_1,\pi_2} = 15/32 < 1/2$.

(ii) Consider the $(3, 12)$ partition $\pi_1 = (3, 4, 5)$ for which formula (5.7) is now satisfied for $j = 1$. For $\pi_2 = (1, 5, 6)$ it holds that $Q_{\pi_1,\pi_2} = 7/18 < 1/2$. ◀

Case 7: Consider an (n, σ) partition $\pi_1 = (1^{t_1} 2^{t_2} \ldots)$ such that

$$(\exists\, m',\ i_{m'} \neq 1)\; (Q_{\pi_1,\delta(\pi_1,m')} < Q_{\pi_1,\delta(\pi_1,m)}) , \tag{5.8}$$

where m is a max-decrement position as defined in (5.6). We can safely assume that (5.7) does not hold as that case was covered before.

First assume there exists an m satisfying (5.6) and for which $t_{i_m-1} \neq 0$, together with an m' satisfying (5.8). Furthermore, assume that it holds that $i_{m'} \neq i_m - 1$ or $t_{i_{m'}} > 1$. We need at least one of these two conditions to hold because otherwise it is impossible to increment an occurrence of $i_m - 1$ and decrement an occurrence of $i_{m'}$. So, we are able to construct π_2 starting from π_1 by incrementing an occurrence of $i_m - 1$ and decrementing $i_{m'}$. Noting that $t_{i_m-1} + t_{i_m} > t_{i_{m'}-1} + t_{i_{m'}}$ (due to (5.8)), we obtain that

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_{i_m-1} + t_{i_m} - t_{i_{m'}} - t_{i_{m'}-1}}{2n^2} < \frac{1}{2}\,.$$

Secondly, let us assume there exists an m satisfying (5.6) and for which $t_{i_m-1} = 0$, together with an m' satisfying (5.8). We build π_2 starting from π_1, by incrementing i_m and decrementing $i_{m'}$. Noting that $t_{i_m} > t_{i_{m'}-1} + t_{i_{m'}}$, we now obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_{i_m} - t_{i_{m'}} - t_{i_{m'}-1}}{2n^2} < \frac{1}{2}\,.$$

Note that t_{i_m+1} must equal 0, as m is a max-decrement position and $t_{i_m-1} = 0$, and therefore we can safely omit it in the above expression.

It is easy to see that the only partitions not covered by the previous assumptions while satisfying (5.8) and not satisfying (5.7), correspond to the following two types of partitions:

$$\pi_1 = (1^{t_1}3^1 4^{t_4}) \quad \text{, with } t_1 > 0 \text{ and } t_4 > 0\,, \tag{5.9}$$
$$\pi_1 = (1^{t_1}2^1 3^{t_3}) \quad \text{, with } 0 \le t_1 < t_3\,. \tag{5.10}$$

Indeed, it must hold that $i_{m'}$ and i_m are unique, that $i_{m'} = i_m - 1$, $t_{i_{m'}} = 1$ and that (5.7) is not satisfied. There must only be one possible choice for $i_{m'}$ and i_m as there otherwise would exist a choice such that $i_{m'} \neq i_m - 1$. The fact that i_m and $i_{m'}$ are unique implies that $\#\{i \mid i > 1 \wedge t_i \neq 0\} = 2$. Furthermore, the fact that $i_m - 1 = i_{m'}$ and that (5.7) is not satisfied, imply that either $m = 4$ and $t_1 \neq 0$, or $m = 3$ and $t_1 < t_3$.

First, consider partitions of type (5.9). If $t_1 > 1$, then we make π_2 from π_1 by incrementing an occurrence of 1 and decrementing 3, obtaining

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_1 - 1}{2n^2} < \frac{1}{2}\,.$$

Now, suppose $t_1 = 1$. Unless there is only one occurrence of 4, which corresponds to Case 4, we can construct π_2 from π_1 by decrementing 3 and incrementing an occurrence of 4. We then obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_4 - 1}{2n^2} < \frac{1}{2}\,.$$

Secondly, let us consider partitions of type (5.10). When $t_1 < t_3 - 1$, the partition $\pi_2 = (1^{t_1+1}3^{t_3-1}4^1)$ wins from π_1, since

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_3 - t_1 - 1}{2n^2} < \frac{1}{2}\,.$$

Suppose now that $t_1 = t_3 - 1$. If $t_1 = 0$, then $\pi_1 = (2,3)$ belongs to the class of partitions covered in Case 2. If $t_1 > 0$, then the partition π_1 is of type $(1^{l-1}2^1 3^l)$, with $l > 1$, and these are all optimal strategies. Indeed, using increment/decrement operations, we can transform the $(2l, 4l+1)$ partition $\pi_1 = (1^{l-1}2^1 3^l)$ into any other $(2l, 4l+1)$ partition. First note that an increment of 2 is useless, as this would cancel out an earlier increment/decrement operation. Therefore, for any partition π_1'', obtained as an intermediate step by an increment/decrement operation performed on partition π_1' in this increment/decrement process, it holds that $Q_{\pi_1,\pi_1'} \geq Q_{\pi_1,\pi_1''}$ and as $Q_{\pi_1,\pi_1} = 1/2$ we obtain that π_1 is indeed an optimal strategy. As it is easily verified that for $(n, \sigma) = (2l, 4l+1)$ the diophantine systems (5.4) and (5.5) have no solution, it follows from Theorem 5.4.1 (of which a part of the proof still needs to be given below), that $(1^{l-1}2^1 3^l)$ is the only optimal strategy of the $(2l, 4l+1)$-game, with $l > 1$. ■

EXAMPLE – 5.5.4:

(i) Consider the $(6, 23)$ partition $\pi_1 = (1, 2, 3, 5, 6, 6)$, $m = 5$ or $m = 6$, $t_{i_m - 1} = 1$, then the possible values for m' are 2 and 3. Choosing π_2 to be one of the partitions $(1, 1, 3, 6, 6, 6)$ and $(1, 2, 2, 6, 6, 6)$, we obtain $Q_{\pi_1,\pi_2} = 35/72$.

(ii) Consider the $(4, 11)$ partition $\pi_1 = (1, 3, 3, 4)$, $m = 4$, $i_{m'} = i_m - 1$, $t_{i_{m'}} = 2 > 1$. If we choose $\pi_2 = (1, 2, 4, 4)$, then $Q_{\pi_1,\pi_2} = 15/32$.

(iii) Consider the $(4, 12)$ partition $\pi_1 = (1, 3, 4, 4)$. For $\pi_2 = (1, 2, 4, 5)$, we find $Q_{\pi_1,\pi_2} = 15/32$.

(iv) Consider the $(3, 8)$ partition $\pi_1 = (2, 3, 3)$. If we choose $\pi_2 = (1, 3, 4)$, then $Q_{\pi_1,\pi_2} = 4/9$.

◀

5.5.4 Investigation of the remaining dice games

Case 8: If (5.7) and (5.8) are not satisfied, then the partition π_1 should satisfy the following property, for some fixed $C \in \mathbb{N}_0$:

$$\begin{cases} i > 1 \wedge t_i > 0 \Rightarrow t_{i-1} + t_i = C\,, \\ (\forall\, i < i_n)(t_i + t_{i+1} > 0)\,. \end{cases} \tag{5.11}$$

The first property holds because (5.8) is not satisfied, while the second property holds because (5.7) is not satisfied.

The remaining cases for π_1 are therefore of one of the following types ($a, b \in$

$\mathbb{N}$, $a+b>0$, $k,k' \in \mathbb{N}_0$, $k<k'$):

$$(1^a 2^b \dots (2k)^b (2k+1)^a (2k+3)^{a+b} (2k+5)^{a+b} \dots (2k'+1)^{a+b}) \quad (5.12)$$

$$(1^a 2^b \dots (2k-1)^a (2k)^b (2k+2)^{a+b} (2k+4)^{a+b} \dots (2k')^{a+b}) \quad (5.13)$$

$$(1^a 2^b \dots (2k-2)^b (2k-1)^a) \quad (5.14)$$

$$(1^a 2^b \dots (2k-1)^a (2k)^b) \quad (5.15)$$

$$(1^a 3^b 5^b \dots (2k+1)^b) \quad (5.16)$$

To assure that these five cases are mutually exclusive, the following conditions on a and b must be imposed. For type (5.12) and (5.13), $a \neq 0$ and $b \neq 0$ must hold because else π_1 would correspond to type (5.14) or (5.15), or (5.7) would hold. For type (5.15) it must hold that $a \neq 0$ and $b \neq 0$ (making (5.14) and (5.15) mutually exclusive). For type (5.16) it should hold that $a \neq b$, $a \neq 0$ and $b \neq 0$ in order to make it mutually exclusive with (5.14) and to exclude the partitions considered already in Cases 5 and 6.

Subcase 8.1: Suppose π_1 is of type (5.12) or (5.13), with $a \neq 0$ and $b \neq 0$.

Let $\nu = \min\{i \mid t_i = 0\}$. Clearly $\nu > 2$ and $t_{\nu+1} > 1$. Decrement an occurrence of $\nu+1$ by 2, increment another occurrence of $\nu+1$ by 1 and increment an occurrence of 1 by one. The resulting (n,σ) partition π_2 clearly wins from π_1. Indeed, for case (5.12) we obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b}{2n^2} < 1/2\,,$$

while for case (5.13) we obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{a}{2n^2} < 1/2\,.$$

■

EXAMPLE – 5.5.5:

(i) Consider the $(12,39)$ partition $\pi_1 = (1^2 2^3 3^2 5^5)$. For the partition $\pi_2 = (1^1 2^4 3^3 5^3 6^1)$, it holds that $Q_{\pi_1,\pi_2} = 141/288$.

(ii) Consider the $(12,58)$ partition $\pi_1 = (1^1 2^2 3^1 4^2 6^3 8^3)$. For the partition $\pi_2 = (2^3 3^1 4^3 6^1 7^1 8^3)$, we find $Q_{\pi_1,\pi_2} = 143/288$. ◀

Subcase 8.2: Suppose π_1 is of type (5.16), with $a \neq b$, $a \neq 0$ and $b \neq 0$.

Let us first consider $a > b$. We construct π_2 from π_1 by incrementing an occurrence of 1 and decrementing an occurrence of 3 to obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{a-b}{2n^2} < \frac{1}{2}\,.$$

Next, suppose $a < b$. If $b > 2$, then we construct π_2 from π_1 by decrementing an occurrence of 3 by two and incrementing two other occurrences of 3 by one. We obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b-a}{2n^2} < \frac{1}{2}\,.$$

When $0 < a < b = 2$ and $n > 3$, we construct π_2 from π_1 by decrementing an occurrence of 3 by two, incrementing the other occurrence of 3 by one and incrementing an occurrence of 5 by one and obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b-a}{2n^2} < \frac{1}{2}\,.$$

The case $a = 1$, $b = 2$ and $n = 3$ corresponds to Case 3. ∎

EXAMPLE – 5.5.6:

(i) Consider the $(7, 19)$ partition $\pi_1 = (1^3 3^2 5^2)$. Choosing $\pi_2 = (1^2 2^2 3^1 5^2)$ yields $Q_{\pi_1,\pi_2} = 24/49$.

(ii) Consider the $(4, 10)$ partition $\pi_1 = (1^1 3^3)$. For $\pi_2 = (1^2 4^2)$, it holds that $Q_{\pi_1,\pi_2} = 7/16$.

(iii) Consider the $(5, 17)$ partition $\pi_1 = (1^1 3^2 5^2)$. Choosing $\pi_2 = (1^2 4^1 5^1 6^1)$, it follows that $Q_{\pi_1,\pi_2} = 12/25$. ◀

Subcase 8.3: Suppose π_1 is of type (5.14) or (5.15) (with $a \neq 0$ and $b \neq 0$ in case of (5.15)), which implies that

$$(\forall\, i < i_n)(t_i + t_{i+1} = C = a + b)\,. \tag{5.17}$$

We will prove that such (n, σ) partitions do not lose from any other (n, σ) partition. Consider an (n, σ) partition $\pi_2 = (j_1, \ldots, j_n) = (1^{t'_1} 2^{t'_2} \ldots)$. As π_1 is also an (n, σ) partition, we can obtain π_2 from π_1 step by step using increment/decrement operations on the elements of the intermediate partitions.

When $j_n \leq i_n$ we can obtain π_2 from π_1 gradually by repeatedly incrementing some $k \in \pi_i$ with $k < i_n$ in the intermediate partition π_i and decrementing another l in π_i until partition π_2 is obtained. It is obvious that after every increment/decrement operation, obtaining an intermediate partition π_i, it holds that $Q_{\pi_1,\pi_i} = Q_{\pi_1,\pi_1}$. Indeed, consider such an intermediate partition π_i and let k (resp. l) be the number to be incremented (resp. decremented). From (5.17) it follows that $t_{l-1} + t_l = C = t_k + t_{k+1}$ (recall that $k < \mu$). Let π'_i be the partition obtained from π_i after the mentioned increment/decrement operation. We then obtain

$$Q_{\pi_1,\pi'_i} = Q_{\pi_1,\pi_i} + \frac{t_l + t_{l-1} - t_k - t_{k+1}}{2n^2} = Q_{\pi_1,\pi_i}\,.$$

Since the end result of the transformation is π_2 and we started from π_1, we obtain $Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_i} = Q_{\pi_1,\pi_1} = 1/2$.

When $j_n > i_n$ we can use increment/decrement operations to obtain π_2 from π_1 while only decrementing numbers $l \leq i_n$. There will be at least one increment/decrement operation that decrements a number $l \leq i_n$ and increments $k = i_n$. Therefore, it holds for case (5.14) that $Q_{\pi_1,\pi_2} \geq Q_{\pi_1,\pi_1} + b/(2n^2)$ and for case (5.15) that $Q_{\pi_1,\pi_2} \geq Q_{\pi_1,\pi_1} + a/(2n^2)$. This proves that the (n, σ)

partition π_1 does not lose from any (n,σ) partition and therefore π_1 is an optimal strategy. ∎

EXAMPLE – 5.5.7:

(i) Consider the $(12,36)$ partition $\pi_1 = (1^2 2^3 3^2 4^3 5^2)$, which is of type (5.14), and the $(12,36)$ partition $\pi_2 = (1^2 2^1 3^4 4^5)$. Using increment/decrement operations, we can transform π_1 into π_2:

$$\pi_1 = (1^2 2^3 3^2 4^3 5^2) \to \pi_1' = (1^2 2^2 3^3 4^4 5^1) \to \pi_1'' = \pi_2 = (1^2 2^1 3^4 4^5)\,.$$

It holds that $Q_{\pi_1,\pi_1} = 1/2 = Q_{\pi_1,\pi_1'} = Q_{\pi_1,\pi_1''} = Q_{\pi_1,\pi_2}$.

(ii) Consider the $(10,26)$ partition $\pi_1 = (1^2 2^3 3^2 4^3)$, which is of type (5.15), and the $(10,26)$ partition $\pi_2 = (1^3 2^4 5^3)$. We again transform π_1 into π_2:

$$\begin{aligned}\pi_1 = (1^2 2^3 3^2 4^3) \to \pi_1' = (1^3 2^2 3^2 4^2 5^1) \to \pi_1'' = (1^3 2^3 3^1 4^1 5^2)\\ \to \pi_1''' = \pi_2 = (1^3 2^4 5^3)\,.\end{aligned}$$

Since $Q_{\pi_1,\pi_1'} = 51/100 > 1/2$, we obtain $Q_{\pi_1,\pi_2} > 1/2$. ◀

All possible (n,σ) partitions have been considered in the above cases and the previously obtained results already show how the optimal strategies look. We still need to prove, for (n,σ) games with $n > 2$, that the existence of partitions of type (5.14), resp. (5.15), is equivalent to condition (5.4), resp. (5.5), from Theorem 5.4.1 and obtain the number of optimal strategies for (n,σ) games of one of these two types. As was already mentioned, the first three conditions of Theorem 5.4.1 correspond to the special Cases 1–4 considered in this subsection. The fourth condition of Theorem 5.4.1 was obtained in Case 7.

5.5.5 Finalizing the proof of Theorem 5.4.1

We will now prove, for $n > 2$, that condition (5.4), resp. (5.5), is equivalent to the condition that there exists at least one (n,σ) partition π_1 that is of type (5.14), resp. (5.15), where for the case of (5.15) it is required that $a \neq 0$ and $b \neq 0$. It is obvious that case (5.14) implies $n = (a+b)k - b$ and that case (5.15) implies $n = (a+b)k$. For case (5.14) we obtain as sum

$$\begin{aligned}\sigma &= a\,(1+3+\ldots+(2\,k-1)) + b\,(2+4+\ldots+2\,k-2)\\ &= a\,k^2 + b\,(k-1)\,k = (a+b)\,k^2 - b\,k = n\,k\,,\end{aligned}$$

while for case (5.15) we obtain as sum

$$\begin{aligned}\sigma &= a\,(1+3+\ldots+(2\,k-1)) + b\,(2+4+\ldots+2\,k)\\ &= a\,k^2 + b\,(k+1)\,k = (a+b)\,k^2 + b\,k = (n+b)\,k\,.\end{aligned}$$

The above proves that if π_1 is an optimal strategy, one of the five conditions from Theorem 5.4.1 is satisfied. On the other hand, it is obvious that whenever one of those five conditions is satisfied, there exists an optimal strategy, which concludes the proof of Theorem 5.4.1. The above reasoning also proves the statements in Propositions 5.4.3 and 5.4.5 about the multiplicity representation of the optimal strategies.

We still need to prove that conditions (5.4) and (5.5) are mutually exclusive. Therefore, suppose there exist a, b, k satisfying (5.4) and a', b', k' satisfying (5.5) (adding accents where appropriate). We then have that $(n + b')k' = \sigma = nk$, which implies $b'k' = n(k - k')$. As $n = (a' + b')k'$, it follows that $n = a'k' + n(k - k')$ and either $k = k'$ which implies $b' = 0$, or $k = k' + 1$ which implies $a' = 0$. But both $a' = 0$ and $b' = 0$ were excluded in (5.5) and both conditions are therefore mutually exclusive.

5.5.6 Finalizing the proof of Proposition 5.4.3

To completely prove Proposition 5.4.3, we must still determine the number of optimal strategies in a game of type (5.4).

Assume a, b, k are solutions of (5.4), with $k > 1$ ($k = 1$ corresponds to Case 5) and with k being invariant for all solutions as this follows immediately from (5.4). Suppose now that $n = (a + b)\,k - b$ and $n = (a' + b')\,k - b'$. It follows that $(a' - a)\,k = (b - b')\,(k - 1)$. Since these are all integers, this is equivalent to $a' = a + l\,(k - 1)$ and $b' = b - l\,k$, for some integer l. Restricting a' and b' such that $a' \geq 0$ and $b' \geq 0$ we obtain that l can vary from $-\lfloor a/(k-1) \rfloor$ to $\lfloor b/k \rfloor$ and there are indeed exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ solutions.

5.5.7 Finalizing the proof of Proposition 5.4.5

Finally, the proof of Proposition 5.4.5 is concluded by showing that there is exactly one optimal strategy in a game of type (5.5). There exist a, b, k for which $n = (a + b)\,k$, $\sigma = (n + b)\,k$, $a \neq 0$ and $b \neq 0$. Suppose $\pi_1 = (i_1, \ldots, i_n)$ and $\pi_2 = (i'_1, \ldots, i'_n)$ are two different optimal strategies. As was shown above, both partitions must be of type (5.15), with a and b different from 0.

Suppose $i_n < i'_n$. Using increment/decrement operations we can construct π_2 from π_1, only decrementing numbers smaller than or equal to i_n, and in at least one of these intermediate steps, transforming the intermediate partition π_i to π'_i, the number i_n will be incremented and we therefore obtain $Q_{\pi_1,\pi_2} \geq Q_{\pi_1,\pi'_i} = Q_{\pi_1,\pi_i} + a/(2n^2) > Q_{\pi_1,\pi_i} \geq Q_{\pi_1,\pi_1} = 1/2$, implying $Q_{\pi_1,\pi_2} > 1/2$ and π_2 is then not an optimal strategy. The case $i_n > i'_n$ is completely analogous. Therefore, k is an invariant of the solution space of (5.15), which implies that the values of a and b are also fixed. Thus, there exists only one optimal strategy in games of type (5.15).

As mentioned before, it is obvious that Proposition 5.4.3 (resp. Proposition 5.4.5) implies Corollary 5.4.7 (resp. Corollary 5.4.8). This completes the proof of the theorem, propositions and corollaries of Section 5.4.

The Law of Fives: All things happen in Fives, or are
divisible by or are multiples of Five, or are somehow
directly or indirectly appropriate to 5.
(The Law of Fives is never wrong)
— COMMON SENSE

The truth is five but men have only one name for it.
— PATAMUNZO LINGANANDA

6 Comparing Coupled Random Variables

In Chapter 4 the generalized dice model was introduced and an emphasis was laid upon comparing independent random variables. We now investigate some dice models in which the random variables are coupled differently. The first two sections are concerned with discrete dice models, while the third section discusses specific continuous dice models. In the first section a reformulation of the probabilistic relation $Q = [q_{ij}]$ (defined in (4.1)) in function of the bivariate c.d.f. is introduced and a general transformation algorithm that can be used to simplify the determination of the characteristic transitivity of certain models is obtained. In the same section, two generalized discrete dice models, denoted as the discrete dice$_{\mathbf{M}}$ model (resp. discrete dice$_{\mathbf{L}}$ model), in which the random variables are pairwisely coupled by the $T_{\mathbf{M}}$-copula (resp. $T_{\mathbf{L}}$-copula) are discussed and alternative representations for a specific generic class of these dice models is obtained. Chapter 7 will introduce games that are played in such 2-dimensional models. In Section 2, the characteristic transitivity of both models is obtained. Section 3 then discusses the continuous dice$_{\mathbf{M}}$ and dice$_{\mathbf{L}}$ models are discussed and it is proven that the characteristic transitivity remains unchanged w.r.t. the corresponding discrete dice models. Most of the results from this chapter can be found in [22, 24, 29].

6.1 Alternative representation for the specific models

This section is concerned with discrete dice models and most of the results can be found in [29]. It is well known that, for discrete random variables X_i and X_j, $p_{X_i,X_j}(k,l)$ can be obtained from F_{X_i,X_j} as

$$p_{X_i,X_j}(k,l) = F_{X_i,X_j}(k,l) + F_{X_i,X_j}(k-1,l-1) - F_{X_i,X_j}(k,l-1) - F_{X_i,X_j}(k-1,l)\,. \tag{6.1}$$

6.1.1 The diagonal formula

We begin this subsection with proving an alternative representation for the probabilistic relation $Q = [q_{ij}]$ generated by a discrete dice model.

PROPOSITION – 6.1.1: *Let X_i and X_j be 2 discrete random variables coupled by the copula C. It then holds that $q_{ij} = \mathrm{Prob}(X_i > X_j) + \frac{1}{2}\mathrm{Prob}(X_i = X_j)$ can be rewritten as*

$$q_{ij} = \frac{1}{2}\left(1 + \sum_k \Big(C(F_{X_i}(k), F_{X_j}(k-1)) - C(F_{X_i}(k-1), F_{X_j}(k))\Big)\right). \tag{6.2}$$

This is called the diagonal formula.

Proof:

$$
\begin{aligned}
q_{ij} &= \sum_k \sum_{l<k} \mathrm{Prob}(X_i = k \wedge X_j = l) + \frac{1}{2}\sum_k \mathrm{Prob}(X_i = k \wedge X_j = k) \\
&= \sum_k \Big(C(F_{X_i}(k), F_{X_j}(k-1)) - C(F_{X_i}(k-1), F_{X_j}(k-1)) \Big) + \\
&\quad \frac{1}{2}\sum_k \Big(C(F_{X_i}(k), F_{X_j}(k)) + C(F_{X_i}(k-1), F_{X_j}(k-1)) - \\
&\quad C(F_{X_i}(k-1), F_{X_j}(k)) - C(F_{X_i}(k), F_{X_j}(k-1)) \Big) \\
&= \frac{1}{2}\left(1 + \sum_k \Big(C(F_{X_i}(k), F_{X_j}(k-1)) - C(F_{X_i}(k-1), F_{X_j}(k)) \Big) \right).
\end{aligned}
$$

■

The diagonal formula can be used to prove the following lemma.

LEMMA – 6.1.2: *Let X_i, X_j, X_i' and X_j' be discrete random variables, and let $p_{X_j}(l) = 0$, for some $l \in \mathbb{Z}$. Let $p_{X_i'}(k) = p_{X_i}(k)$ and $p_{X_j'}(k) = p_{X_j}(k)$ for $k < l$, let $p_{X_i'}(k+1) = p_{X_i}(k)$ and $p_{X_j'}(k+1) = p_{X_j}(k)$ for $k > l$ and let $p_{X_i'}(l) + p_{X_i'}(l+1) = p_{X_i}(l)$ and $p_{X_j'}(l) = p_{X_j'}(l+1) = 0$. It then holds for any copula C that $Q(X_i, X_j) = Q(X_i', X_j')$.*

Proof:

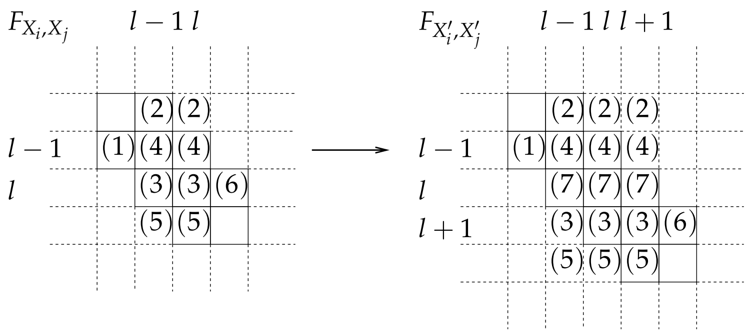


Figure 6.1: *Transformation illustration.*

In Figure 6.1 we have that, e.g.,

$$
(1) = C(F_{X_i'}(l-1), F_{X_j'}(l-2)),
\qquad
\begin{aligned}
(2) &= C(F_{X_i'}(l-2), F_{X_j'}(l-1)) \\
&= C(F_{X_i'}(l-2), F_{X_j'}(l)) \\
&= C(F_{X_i'}(l-2), F_{X_j'}(l+1)).
\end{aligned}
$$

We need to prove that $Q(X'_i, X'_j) - Q(X_i, X_j) = 0$, which due to the diagonal formula is equivalent to

$$\begin{aligned}&\Big(C(F_{X'_i}(l+1), F_{X'_j}(l)) - C(F_{X'_i}(l), F_{X'_j}(l+1)) +\\ &C(F_{X'_i}(l), F_{X'_j}(l-1)) - C(F_{X'_i}(l-1), F_{X'_j}(l))\Big) -\\ &\Big(C(F_{X_i}(l), F_{X_j}(l-1)) - C(F_{X_i}(l-1), F_{X_j}(l))\Big) = 0\,,\end{aligned}$$

which is equivalent to $((3) - (7) + (7) - (4)) - ((3) - (4)) = 0$, using the notations from Figure 6.1, and this is trivially satisfied. ∎

The above lemma then implies the following proposition.

PROPOSITION – 6.1.3: *The probabilistic relation generated by a generalized discrete dice model in which each random variable X_i is uniformly distributed over a multiset A_i such that all multisets are mutually disjoint can also be generated by a generalized discrete dice model in which the same copulas are used to couple the random variables and in which each random variable X'_i is uniformly distributed over a set A'_i, all sets of equal cardinality and again mutually disjoint.*

Proof:
We give the proof for such a dice model consisting of 2 random variables, the generalization to more than 2 random variables is straightforward. Let X_1 and X_2 be the random variables of the model. Transform X_1 and X_2 into X'_1 and X'_2 step by step, by replacing each element of A_1 (resp. A_2) by $\#A_1$ (resp. $\#A_2$) consecutive integers in an order preserving way (starting with the integer 1) and such that the obtained sets remain disjunct. Due to Proposition 6.1.2 and the fact that the multisets are mutually disjoint, we have that $Q(X_1, X_2) = Q(X'_1, X'_2)$. ∎

Note that the above proposition can be seen as a generalization of Theorem 3.3.3, however only for disjoint multisets. A general algorithm to transform two nondisjoint multisets into two disjoint multisets while preserving the corresponding probabilistic relation (and using the same copula) cannot be given. We now illustrate the above proposition with an example.

EXAMPLE – 6.1.4:
Consider the random variables X_1 and X_2 uniformly distributed over the multiset $A_1 = \{1, 2, 4, 4, 4, 6\}$, resp. $A_2 = \{3, 7, 8\}$. Using the short notation $F(k, l) = F_{X_1,X_2}(k, l) = C(F_{X_1}(k), F_{X_2}(l))$, it then holds that

$$\begin{array}{lll}
F(k,k-1) = F(k-1,k) = 0\,,\ \forall\, k < 3\,, & F(k,k-1) = F(k-1,k) = 1\,,\ \forall\, k > 8\,, & \\
F(3,2) = 0 & ,\ F(2,3) = C(1/3,1/3) & ,\ F(4,3) = C(5/6,1/3)\,,\\
F(3,4) = C(1/3,1/3) & ,\ F(5,4) = C(5/6,1/3) & ,\ F(4,5) = C(5/6,1/3)\,,\\
F(6,5) = C(1,1/3) & ,\ F(5,6) = C(5/6,1/3) & ,\ F(7,6) = C(1,1/3)\,,\\
F(6,7) = C(1,2/3) & ,\ F(8,7) = C(1,2/3) & ,\ F(7,8) = C(1,1)\,.
\end{array}$$

Using the diagonal formula (6.2), we then obtain that

$$Q(X_1, X_2) = 1/3 - C(1/3, 1/3)\,.$$

Let now X'_1 and X'_2 be random variables uniformly distributed over the multisets A'_1, resp. A'_2, defined by

$$\begin{aligned} A'_1 &= \mathbb{N}[1,6] \cup \mathbb{N}[13,24]\,,\\ A'_2 &= \mathbb{N}[7,12] \cup \mathbb{N}[25,36]\,. \end{aligned} \tag{6.3}$$

Note that A'_1 and A'_2 are obtained from A_1 and A_2 using the algorithm from Proposition 6.1.3. More precisely, 1 is replaced by $\{1,2,3\}$, 2 by $\{4,5,6\}$, 3 by $\{7,8,9,10,11,12\}$, the three occurrences of 4 by $\mathbb{N}[13,24]$, and so on. Using the short notation $F'(k,l) = F_{X'_1,X'_2}(k,l) = C(F_{X'_1}(k), F_{X'_2}(l))$, we now obtain

$$\begin{array}{ll} F'(k+1,k) = F'(k-1,k) = 0 & , \forall\, k < 7\,,\\ F'(k+1,k) = F'(k-1,k) = C(1/3,(k-6)/18) & , \forall\, k \in \mathbb{N}[7,11]\,,\\ F'(k,k-1) = F'(k,k+1) = C((k-6)/18,1/3) & , \forall\, k \in \mathbb{N}[13,23]\,,\\ F'(k+1,k) = F'(k-1,k) = (k-18)/18 & , \forall\, k \in \mathbb{N}[25,36]\,,\\ F'(k+1,k) = F'(k-1,k) = 1 & , \forall\, k > 36\,,\\ F'(11,12) = F'(12,13) = C(1/3,1/3)\,, & \\ F'(24,23) = F'(25,24) = 1/3\,. & \end{array}$$

It therefore holds that

$$2Q(X'_i, X'_j) - 1 = -1/3 - 2C(1/3, 1/3)\,. \tag{6.4}$$

From the above calculations it indeed follows that $Q(X_1, X_2) = Q(X'_1, X'_2)$. Note that the right-hand side in (6.4) is the sum

$$\sum_k \left(C(F_{X_1}(k), F_{X_2}(k-1)) - C(F_{X_1}(k-1), F_{X_2}(k)) \right)\,,$$

and not, e.g., the sum

$$\sum_k \left(C(F_{X_1}(k), F_{X_2}(k-1)) - C(F_{X_1}(k), F_{X_2}(k+1)) \right)\,,$$

which is in accordance with the diagonal formula. Both sums can differ, as is the case in this example. ◀

The above example clearly illustrates that the diagonal formula is very useful to obtain the probabilistic relation Q in function of an unknown copula C (for discrete r.v.). The next proposition introduces an alternative representation of discrete r.v. with finite support and rational probabilities. This representation will prove useful in the next subsections.

PROPOSITION – 6.1.5: *Any collection of discrete random variables for which the probability masses only take rational values and which all have finite support, can be regarded as a specific collection of random variables, each uniformly distributed over the elements of a multiset of integers (one multiset per random variable), with all multisets having equal cardinality.*

Proof:
Let $(X_1, X_2, \ldots, X_m)$ be a collecton of discrete random variables, for which $p_{X_j}(i) = a_{ji}/b_{ji}$, with a_{ji} and b_{ji} coprime, for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}[1,m]$, and let b be the least common multiple of the denominators b_{ji}. Consider the multisets A_j consisting of $a_{ji}b$ occurrences of i, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}[1,m]$, and the discrete random variables X'_j uniformly distributed over the elements of A_j, for all $j \in \mathbb{N}$. It then holds that $p_{X_j}(i) = p_{X'_j}(i)$, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}[1,m]$. ■

We now give a short illustration of the above proposition.

EXAMPLE – 6.1.6:
Consider the discrete random variables X_1 and X_2, for which $p_{X_1}(5) = 2/3$, $p_{X_1}(6) = 1/3$, $p_{X_2}(3) = 1/5$ and $p_{X_2}(4) = p_{X_2}(6) = 2/5$. We now construct the multisets A_1 and A_2 using the algorithm from the proof of the above proposition, thus obtaining $A_1 = \{5,5,5,5,5,5,5,5,5,5,6,6,6,6,6\}$ and $A_2 = \{3,3,3,4,4,4,4,4,4,6,6,6,6,6,6\}$. Note that both multisets have the same cardinality 15. X'_1 and X'_2 are then defined as discrete random variables uniformly distributed on A_1 resp. A_2. It then holds that $p_{X_1}(i) = p_{X'_1}(i)$ and $p_{X_2}(i) = p_{X'_2}(i)$, for any $i \in \mathbb{Z}$. ◄

In the following two subsections, we will consider discrete random variables with rational probability masses and finite support and we will use the multiset representation as introduced in Proposition 6.1.5.

6.1.2 The discrete dice$_\mathbf{M}$ model

A dice$_\mathbf{M}$ model is a dice model in which all r.v. are artificially coupled by the $T_\mathbf{M}$-copula. We discuss in this subsection in particular the class of discrete dice$_\mathbf{M}$ models in which the r.v. have finite support and rational probability masses. As has been said before, the probabilistic relation $Q = [q_{ij}]$ is, for discrete r.v., defined by

$$q_{ij} = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k=l} p_{X_i,X_j}(k,l)\,. \tag{6.5}$$

We first obtain a general result for the probabilistic relation generated by any discrete dice$_\mathbf{M}$ model. Using (6.1), we obtain

$$\begin{aligned} p^{\mathbf{M}}_{X_i,X_j}(k,l) \;=\; & \min(F_{X_i}(k), F_{X_j}(l)) + \min(F_{X_i}(k-1), F_{X_j}(l-1)) - \\ & \min(F_{X_i}(k), F_{X_j}(l-1)) - \min(F_{X_i}(k-1), F_{X_j}(l))\,, \end{aligned}$$

which is equivalent to

$$p^{\mathbf{M}}_{X_i,X_j}(k,l) = \begin{cases} 0 & \text{, if } F_{X_i}(k) \leq F_{X_j}(l-1) \vee F_{X_j}(l) \leq F_{X_i}(k-1)\,, \\ \min(F_{X_i}(k), F_{X_j}(l)) - \max(F_{X_i}(k-1), F_{X_j}(l-1)) & \\ & \text{, otherwise}\,. \end{cases} \tag{6.6}$$

When representing the r.v. X_i as uniformly distributed r.v. over multisets of equal cardinality (see Proposition 6.1.5), thereby restricting ourselves to discrete r.v. with finite support and rational probability masses, we obtain the following useful representation of the generated probabilistic relation.

PROPOSITION – 6.1.7: *The probabilistic relation generated by a dice$_{\mathbf{M}}$ model consisting of a collection $(A_1, A_2, \ldots, A_m)$ of m multisets, each multiset having cardinality n, is given by $Q = [q^{\mathbf{M}}_{ij}]$, where*

$$q^{\mathbf{M}}_{ij} = \frac{\#\{k \mid i_k > j_k\}}{n} + \frac{\#\{k \mid i_k = j_k\}}{2n}\,, \tag{6.7}$$

with $A_i = \{i_1, i_2, \ldots, i_n\}$ and $A_j = \{j_1, j_2, \ldots, j_n\}$.

Proof:
As each element in the set has probability $1/n$, the first part of (6.6) is equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_\ell = l\} = 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = 0$. The second part is equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_\ell = l\} = f > 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = f/n$. For these random variables, we can therefore reformulate (6.5) as (6.7). ∎

6.1.3 The discrete dice$_{\mathbf{L}}$ model

A dice$_{\mathbf{L}}$ model is a dice model in which all r.v. are artificially coupled by the $T_{\mathbf{L}}$-copula. Similar to the previous subsection, in this subsection we discuss the class of discrete dice$_{\mathbf{L}}$ models in which the r.v. have finite support and rational probability masses. We again start with a result that holds for any discrete dice$_{\mathbf{L}}$ model, after which r.v. with finite support and rational probability masses are considered.

Using (6.1), we obtain

$$\begin{aligned} p^{\mathbf{L}}_{X_i,X_j}(k,l) = {} & \\ & \max(F_{X_i}(k) + F_{X_j}(l) - 1, 0) + \max(F_{X_i}(k-1) + F_{X_j}(l-1) - 1, 0) \\ & - \max(F_{X_i}(k) + F_{X_j}(l-1) - 1, 0) - \max(F_{X_i}(k-1) + F_{X_j}(l) - 1, 0)\,, \end{aligned}$$

which is equivalent to

$$p^{\mathbf{L}}_{X_i,X_j}(k,l) = \begin{cases} 0\ \text{, if } F_{X_i}(k) \le 1 - F_{X_j}(l) \vee 1 - F_{X_j}(l-1) \le F_{X_i}(k-1)\,, \\ \min(F_{X_i}(k), 1 - F_{X_j}(l-1)) - \max(F_{X_i}(k-1), 1 - F_{X_j}(l)) \\ \quad \text{, otherwise}\,. \end{cases} \tag{6.8}$$

PROPOSITION – 6.1.8: *The probabilistic relation generated by a dice$_{\mathbf{L}}$ model consisting of a collection $(A_1, A_2, \ldots, A_m)$of m multisets, each multiset having cardinality n is given by $Q = [q^{\mathbf{L}}_{ij}]$*

$$q^{\mathbf{L}}_{ij} = \frac{\#\{k \mid i_k > j_{n-k+1}\}}{n} + \frac{\#\{k \mid i_k = j_{n-k+1}\}}{2n}\,, \tag{6.9}$$

with $A_i = \{i_1, i_2, \ldots, i_n\}$ and $A_j = \{j_1, j_2, \ldots, j_n\}$.

Proof:
The first part of (6.8) is equivalent to demanding that when $\#\{\ell \mid i_\ell = k \wedge j_{n+\ell-1} = l\} = 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = 0$. The second part is then equivalent to saying that when $\#\{\ell \mid i_\ell = k \wedge j_{n+\ell-1} = l\} = f > 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = f/n$. For the considered random variables X_i and X_j, (6.5) can therefore be reformulated into (6.9). ∎

EXAMPLE – 6.1.9:

(i) Consider e.g. the nondisjoint sets $A_1 = \{1,2,5,8\}$ and $A_2 = \{2,3,5,6\}$. Figure 6.2 shows graphically which elements of the multisets have to be compared, when the corresponding random variables are coupled by the $T_{\mathbf{M}}$-, $T_{\mathbf{P}}$- and $T_{\mathbf{L}}$-copulas. We obtain $q^{\mathbf{P}}_{12} = (0 + 0.5 + 2.5 + 4)/16 = 7/16$, $q^{\mathbf{M}}_{12} = 0 + 0 + 1/8 + 1/4 = 3/8$ and $q^{\mathbf{L}}_{12} = 0 + 0 + 1/4 + 1/4 = 1/2$.

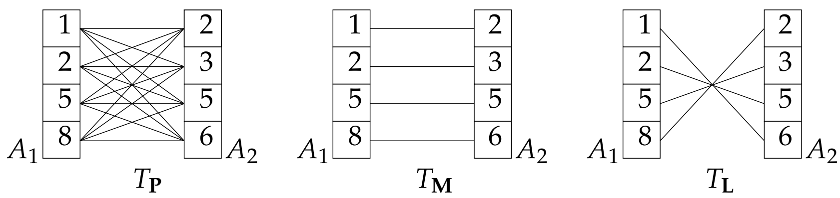


Figure 6.2: *The three comparison methods for a specific example.*

(ii) Reconsider the sets (6.3) from Example 6.1.4. Using (6.7) (resp. (6.9)), we obtain $q^{\mathbf{M}}_{ij} = 0$ (resp. $q^{\mathbf{L}}_{ij} = 1/3$). On the other hand, substituting $C = T_{\mathbf{M}}$ (resp. $C = T_{\mathbf{L}}$) in (6.4) obtains the same result. ◀

6.2 Transitivity of the specific models

6.2.1 Transitivity of discrete dice$_{\mathbf{M}}$ models

PROPOSITION – 6.2.1: *All m-dimensional discrete dice$_{\mathbf{M}}$ models generate $T_{\mathbf{L}}$-transitivite relations.*

Proof:
As follows from Subsection 6.1.2 we only need to consider r.v. uniformly distributed over multisets of equal cardinality (discrete r.v. with infinite support or nonrational probability masses can be approximated as close as possible by r.v. with finite support and rational probability masses). Consider any 3 such multisets $A_i = \{i_1, i_2, \ldots, i_n\}$, $A_j = \{j_1, j_2, \ldots, j_n\}$ and $A_k = \{k_1, k_2, \ldots, k_n\}$. To obtain the probabilistic relation generated by the corresponding dice$_{\mathbf{M}}$ model, it follows from (6.7) that we only need to compare elements of the same order

in the (ordered) multisets, namely all triples (i_l, j_l, k_l), with $l \in \mathbb{N}[1, n]$. It is obvious that the specific comparison done for each such triple contributes at least $1/n$ and at most $2/n$ to the sum $q_{ij} + q_{jk} + q_{ki}$. Summing over the n triples, we obtain $1 \leq q_{ij} + q_{jk} + q_{ki} \leq 2$. ∎

For $m = 3$, the reverse statement is also true.

PROPOSITION – 6.2.2: *Any 3-dimensional $T_{\mathbf{L}}$-transitive probabilistic relation $Q = [q_{ij}]$ with rational elements can be generated by a discrete dice$_{\mathbf{M}}$ model in which the multisets are ordinary mutually disjoint sets.*

Proof:
Let $q_{12} = p/n$, $q_{23} = q/n$, $q_{31} = r/n$ and let (A_1, A_2, A_3) be a standard triplet, with $\#A_1 = \#A_2 = \#A_3 = n$. Furthermore, let $A_1 = \{a_1, a_2, \ldots, a_n\}$, $A_2 = \{a'_1, a'_2, \ldots, a'_n\}$ and $A_3 = \{a''_1, a''_2, \ldots, a''_n\}$, such that $\{a_i, a'_i, a''_i\} = \{3i-2, 3i-1, 3i\}$ for $i \in \mathbb{N}[1, n]$. In order to obtain $Q(A_1, A_2) = q_{12}$ and $Q(A_2, A_3) = q_{23}$, we choose $a_i > a'_i$ for $i \in \mathbb{N}[1, p]$, $a_i < a'_i$ for $i \in \mathbb{N}[p+1, n]$, $a'_i < a''_i$ for $i \in \mathbb{N}[1, n-q]$, $a'_i > a''_i$ for $i \in \mathbb{N}[n-q+1, n]$. It already holds that $a''_i > a_i$ for $i \in \mathbb{N}[p+1, n-q]$ and $a''_i < a_i$ for $i \in \mathbb{N}[n-q+1, p]$. However, for $i \notin \mathbb{N}[p+1, n-q] \cup \mathbb{N}[n-q+1, p]$, it can be chosen freely whether $a''_i > a_i$ or $a''_i < a_i$. As Q is $T_{\mathbf{L}}$-transitive it holds that $n - p - q \leq r \leq 2n - p - q$, and we can therefore choose enough of these remaining i such that $a''_i > a_i$ holds and such that $Q(A_3, A_1) = r/n$, concluding the proof. ∎

It is quite natural to extend the above proposition to any discrete dice$_{\mathbf{M}}$ model.

COROLLARY – 6.2.3: *$T_{\mathbf{L}}$-transitivity is the characteristic transitivity of 3-dimensional discrete dice$_{\mathbf{M}}$ models.*

The question arises whether the reverse property which holds for 3-dimensional $T_{\mathbf{L}}$-transitive probabilistic relations, extends to higher-dimensional relations. The question must be answered negatively, as has been pointed out by Switalski [76] in his analysis of the type of transitivity of the so-called multidimensional model.

DEFINITION – 6.2.4: *The multidimensional model is a preference model in which the preferences generate a probabilistic relation $Q = [q_{ij}]$ defined by*

$$q_{ij} = \sum_{t=1}^{n} \mu_t \, q_{ij}^{(t)}, \tag{6.10}$$

with $q_{ij}^{(t)} \in \{0, 1/2, 1\}$, $q_{ij}^{(t)} = 1 - q_{ji}^{(t)}$ for all $t \in \mathbb{N}[1, n]$ and where the weights μ_t (associated to criterion t) are such that $\mu_t \geq 0$ for all $t \in \mathbb{N}[1, n]$ and $\sum_{t=1}^{n} \mu_t = 1$.

One easily sees that a multidimensional model with all $\mu_i = 1/n$ is equivalent to a dice$_{\mathbf{M}}$ model. Hence, if for a $T_{\mathbf{L}}$-transitive probabilistic relation Q with rational elements a multidimensional model can be constructed that generates

Q, then this model obviously has rational weights μ_t, so that also a collection of ordered lists can be constructed that generates that same probabilistic relation Q by applying the comonotonic comparison strategy. Moreover, if no multidimensional model can be found to generate Q, then also no collection of ordered lists that generates Q exists. This problem has been shown to be closely connected to the coordinate values of the vertices of generalized transitive tournament polytopes [10]. For a review of recent results in this field, the reader is referred to [76]. From Theorem 5.3 of [75], we obtain the following result.

PROPOSITION – 6.2.5: *Let $Q = [q_{ij}]$ be an m-dimensional probabilistic relation with rational elements and $m \leq 5$. Then Q is generated by a collection of ordered lists that are comonotonically compared if and only if Q is $T_{\mathbf{L}}$-transitive.*

Furthermore, it follows from [10] that for every $n > 5$ there exists a $T_{\mathbf{L}}$-transitive probabilistic relation Q that has no representation as in (6.10), and can therefore not be generated by ordered lists that are comonotonically compared.

6.2.2 Transitivity of discrete dice$_{\mathbf{L}}$ models

Partial g-stochastic transitivity was introduced in Definition 2.4.5. In this subsection, this type of transitivity shows up for $g = \min$, and we will denote this specific type of transitivity as partial min-stochastic transitivity.

PROPOSITION – 6.2.6: *All m-dimensional dice$_{\mathbf{L}}$ models generate partial min-stochastic transitivity.*

Proof:
As was the case in the previous subsection, the results from Subsection 6.1.2 imply that we only need to consider multisets of equal cardinality. Consider 3 such multisets $A_i = \{i_1, i_2, \ldots, i_n\}$, $A_j = \{j_1, j_2, \ldots, j_n\}$, $A_k = \{k_1, k_2, \ldots, k_n\}$, and suppose that $q_{ij} > 1/2$, $q_{jk} > 1/2$. If $q_{ik} = 1$ then $q_{ik} \geq \min(q_{ij}, q_{jk})$ holds. Suppose now that $q_{ik} < 1$ and let $p = \min\{l \mid i_l \geq j_{n+1-l}\}$, $q = \min\{l \mid j_l \geq k_{n+1-l}\}$ and $r = \min\{l \mid k_l \geq i_{n+1-l}\}$. First suppose $r > \min(n-p+1, n-q+1)$, it then holds that $q_{ki} \leq (n - \min(n-p+1, n-q+1))/n$, which implies $q_{ik} \geq \min(q_{ij}, q_{jk})$.

Next, suppose $r \leq \min(n-p+1, n-q+1)$. As $\max(p,q) \leq n/2$, it then holds that $i_p \geq j_{n-p+1} \geq j_q \geq k_{n-q+1} \geq k_r \geq i_{n-r+1} \geq i_p$, which implies $i_p = j_{n-p+1} = j_q = k_{n-q+1} = k_r = i_{n-r+1} = i_p$. We therefore have that $j_l = j_q$, for all $q \leq l \leq n-p+1$. First, suppose that $p \geq q$. For $l \in \mathbb{N}[p, n-p]$ it holds that $i_l \geq j_{n-l+1} = j_l = j_q = k_{n-q+1} \geq k_{n-l+1}$, and for $l > n-p$ it holds that (as $q_{ij} > 1/2$) $i_l > j_p \geq j_q = k_{n-q+1} \geq k_{n-l+1}$. It follows that $q_{ik} \geq q_{ij}$. Finally, suppose $p < q$. For $l \in \mathbb{N}[q, n-q]$, it holds that $i_l \geq j_{n-l+1} = j_l \geq k_{n-l+1}$, while for $l > n-q$, it holds that (as $q_{jk} > 1/2$) $i_l \geq i_q \geq j_{n-q+1} > k_q \geq k_{n-l+1}$. It now follows that $q_{ik} \geq q_{jk}$.

Partial stochastic transitivity must therefore be satisfied. ■

PROPOSITION – 6.2.7: *Any 3-dimensional partially min-stochastic transitive probabilistic relation with rational elements can be generated by a discrete dice*$_{\mathbf{L}}$ *model in which the multisets are ordinary mutually disjoint sets.*

Proof:
Let $Q = [q_{ij}]$ be such a probabilistic relation. Note that partial min-stochastic transitivity is unconditional only when at least 2 elements from $\{q_{ij}, q_{jk}, q_{ki}\}$ equal 1/2. We first consider this case. Without loss of generality, suppose $q_{ij} = q_{jk} = 1/2$ and $q_{ki} = a/(2n)$, with $a \in \mathbb{N}[0, n]$. The dice$_{\mathbf{L}}$ model consisting of the following three sets then generates this probabilistic relation.

$$\begin{aligned} A_i &= \mathbb{N}[n+1, n+a] \cup \mathbb{N}[3n+1, 5n-a]\,, \\ A_j &= \mathbb{N}[n+a+1, 2n+a] \cup \mathbb{N}[5n-a+1, 6n-a]\,, \\ A_k &= \mathbb{N}[1, n] \cup \mathbb{N}[2n+a+1, 3n] \cup \mathbb{N}[6n-a+1, 6n]\,. \end{aligned}$$

Suppose now that no such 2 elements equal 1/2, without loss of generality we can assume $\beta_{ijk} > 1/2$ because if that doesn't hold then $\beta_{kji} > 1/2$. As the probabilistic relation has rational elements, we can write them with common denominator n. Suppose first that that the elements can be reordered such that $q_{ij} = c/n$, $q_{jk} = b/n$ and $q_{ki} = a/n$, with $c \geq b > n/2$ and $n - a \geq b$. The dice$_{\mathbf{L}}$ model consisting of the following three sets then generates this probabilistic relation.

$$\begin{aligned} A_i &= \mathbb{N}[n+1+c-a, 2n+c-a]\,, \\ A_j &= \mathbb{N}[1, n-b] \cup \mathbb{N}[n+1, b+c] \cup \mathbb{N}[2n+c-a+1, 3n-a]\,, \\ A_k &= \mathbb{N}[n-b+1, n] \cup \mathbb{N}[b+c+1, n+c-a] \cup \mathbb{N}[3n-a+1, 3n]\,. \end{aligned}$$

Secondly, suppose the elements can be reordered such that $q_{ij} = b/n$, $q_{jk} = c/n$ and $q_{ki} = a/n$, with $c \geq b > n/2$ and $n - a \geq b$. This probabilistic relation is then generated by the dice$_{\mathbf{L}}$ model consisting of the sets

$$\begin{aligned} A_i &= \mathbb{N}[2n-c-a+1, 3n-c-b-a] \cup \mathbb{N}[2n+1, 2n+b]\,, \\ A_j &= \mathbb{N}[1, n-c] \cup \mathbb{N}[3n-c-b+1, 2n] \cup \mathbb{N}[2n+b+1, 3n]\,, \\ A_k &= \mathbb{N}[n-c+1, 2n-c-a] \cup \mathbb{N}[3n-c-b-a+1, 3n-c-b]\,. \end{aligned}$$

As all cases were considered, the proof is concluded. ■

It is quite natural to extend the above proposition to any discrete dice$_{\mathbf{L}}$ model and we thus obtain the following result.

COROLLARY – 6.2.8: *partial min-stochastic transitivity is the characteristic transitivity of 3-dimensional discrete dice*$_{\mathbf{L}}$ *models.*

Again, the question arises whether this inverse statement can be generalized to higher-dimensional probabilistic relations. And again, the question must be answered in negative sense.

PROPOSITION – 6.2.9: *Not all 4-dimensional partially stochastic transitive probabilistic relations (with rational elements) can be generated by a 4-dimensional dice$_{\mathbf{L}}$ model.*

Proof:
The proof is completely similar to the proof of Theorem 6.2.9. Indeed, vital to the proof is that determining the probabilistic relation involves comparing the smallest element of one set to the highest element of the second set. Since the conditions for partial min-stochastic transitivity to be satisfied for the graph of Theorem 6.2.9 are also given by (3.27) and since the dice$_{\mathbf{L}}$ model also involves the comparison of the minimum of one list and the maximum of the other list (as follows from (6.9)), we can conclude the proof. ∎

6.3 Continuous dice$_{\mathbf{M}}$ and dice$_{\mathbf{L}}$ models

For completeness, we end this chapter with considering the investigated dice models for continuous random variables. We therefore investigate the continuous dice$_{\mathbf{M}}$ model, for which the joint c.d.f. is given by

$$F^{\mathbf{M}}_{X_i,X_j}(x,y) = \min(F_{X_i}(x), F_{X_j}(y))\,, \tag{6.11}$$

and the continuous dice$_{\mathbf{L}}$ model, where the joint c.d.f. is given by

$$F^{\mathbf{L}}_{X_i,X_j}(x,y) = \max(0, F_{X_i}(x) + F_{X_j}(y) - 1)\,. \tag{6.12}$$

A nice representation for the generated probabilistic relations in function of the marginal distribution functions can be easily deduced by using a well-known property of the extreme copulas, which we recall in the next proposition [61, 64]. We first need to introduce a definition.

DEFINITION – 6.3.1: *A subset S of $\overline{\mathbb{R}}^2$ is nondecreasing if for any (x,y) and (u,v) in S, $x < u$ implies $y \leq v$. Similarly, a subset S of $\overline{\mathbb{R}}^2$ is nonincreasing if for any (x,y) and (u,v) in S, $x < u$ implies $y \geq v$.*

PROPOSITION – 6.3.2: *For two random variables X_i and X_j it holds that $F_{X_i,X_j} = T_{\mathbf{M}}(F_{X_i}, F_{X_j})$ if and only if the support of F_{X_i,X_j} is a nondecreasing subset of $\overline{\mathbb{R}}^2$. Similarly, it holds that $F_{X_i,X_j} = T_{\mathbf{L}}(F_{X_i}, F_{X_j})$ if and only if the support of F_{X_i,X_j} is a nonincreasing subset of $\overline{\mathbb{R}}^2$.*

We can now reformulate the probabilistic relation generated by continuous dice$_{\mathbf{M}}$ and dice$_{\mathbf{L}}$ models.

PROPOSITION – 6.3.3: *The probabilistic relation $Q = [q^{\mathbf{M}}_{ij}]$ generated by a continuous dice$_{\mathbf{M}}$ model is given by*

$$q^{\mathbf{M}}_{ij} = \int_{x:F_{X_i}(x)<F_{X_j}(x)} f_{X_i}(x)\,dx + \frac{1}{2}\int_{x:F_{X_i}(x)=F_{X_j}(x)} f_{X_i}(x)\,dx\,. \tag{6.13}$$

The probabilistic relation $Q = [q_{ij}^{\mathbf{L}}]$ generated by a continuous dice$_{\mathbf{L}}$ model is given by

$$q_{ij}^{\mathbf{L}} = \int_{x:F_{X_i}(x)+F_{X_j}(x)>1} f_{X_i}(x)\,dx\,. \tag{6.14}$$

Proof:
These results follow directly from Proposition 6.3.2 and they are illustrated in Figure 6.3. The left-hand side of Figure 6.3 represents the support of $F_{X_i,X_j} = \min(F_{X_i}, F_{X_j})$ for the not further defined continuous c.d.f. F_{X_i} and F_{X_j}. From the figure, it is obvious that

$$q_{ij}^{\mathbf{M}} = F_{X_i}(t_1) + (1 - F_{X_i}(t_4)) + \frac{1}{2}(F_{X_i}(t_3) - F_{X_i}(t_2))\,,$$

which is also given by (6.13).

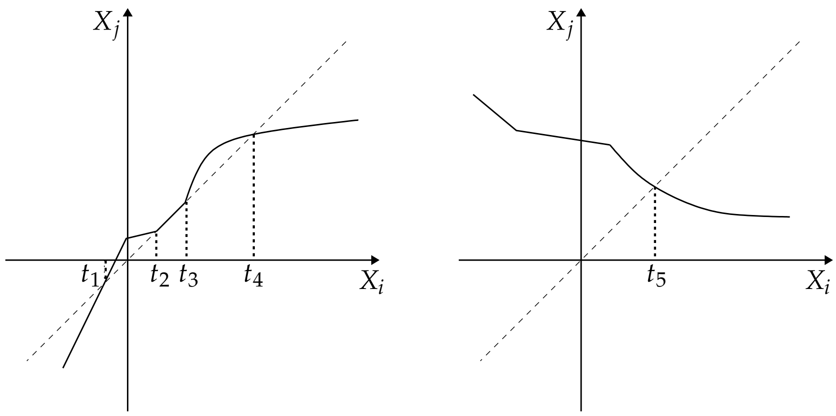


Figure 6.3: *Nondecreasing and nonincreasing supports of F_{X_i,X_j}.*

The right-hand side of Figure 6.3 represents the support of $F_{X_i,X_j} = \max(0, F_{X_i} + F_{X_j} - 1)$ for the not further defined continuous c.d.f. F_{X_i} and F_{X_j}. From the figure, it is obvious that

$$q_{ij}^{\mathbf{L}} = 1 - F_{X_i}(t_5)\,,$$

which is also given by (6.14). ∎

It is interesting to note that, although the random variables X_i and X_j are continuous, their coupling with the $T_{\mathbf{M}}$- or $T_{\mathbf{L}}$-copula is no longer continuous, as left-continuity of F_{X_i,X_j} is not satisfied.

In Figures 6.4 and 6.5 we show a graphical interpretation of (6.13) and (6.14), using the marginal c.d.f. of X_i and X_j. The definition (6.13) comes down to summing the interval lengths of $\{F_{X_i}(x) \mid F_{X_i}(x) < F_{X_j}(x),\, x \in \mathbb{R}\}$ plus one half the interval lengths of $\{F_{X_i}(x) \mid F_{X_i}(x) = F_{X_j}(x),\, x \in \mathbb{R}\}$. For Figure 6.4 we thus obtain

$$q_{ij}^{\mathbf{M}} = t_1 + t_3 + \frac{t_2}{2}\,.$$

The definition (6.14) comes down to summing the interval length of $\{F_{X_i}(x) \mid F_{X_i}(x) + F_{X_j}(x) > 1,\, x \in \mathbb{R}\}$. For Figure 6.5 we then obtain

$$q_{ij}^{\mathbf{L}} = t_4 = 1 - t_5 \,.$$

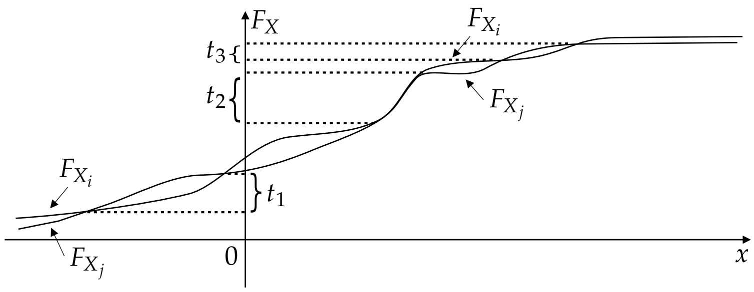


Figure 6.4: *Continuous dice$_{\mathbf{M}}$ model.*

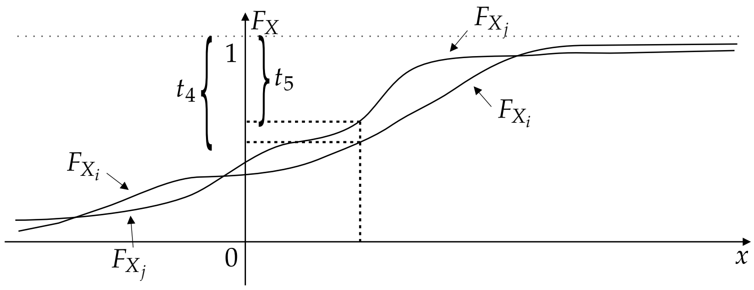


Figure 6.5: *Continuous dice$_{\mathbf{L}}$ model.*

Success is dependent on effort.
— SOPHOCLES

The sun, with all those planets revolving around it
and dependent on it, can still ripen a bunch of grapes
as if it had nothing else in the universe to do.
— GALILEO GALILEI

7 Ordered List Games

The games we introduce in this chapter can be seen as variants of the dice game that was introduced in Chapter 5. The only difference between the games is the copula used for determining the winning probabilities between two dice. Instead of coupling the marginal distribution functions with the $T_{\mathbf{P}}$-copula as was implicitly done in Chapter 5 obtaining independent dice, we now couple them with the $T_{\mathbf{M}}$- or $T_{\mathbf{L}}$-copula. As the independence is lost, it is no longer convenient to think of the random variables as dice that are thrown. Therefore, for clarity, we drop the notion of dice and just consider (n, σ) partitions. The game description is completely analogous to the one in Chapter 5, except that the probabilistic relations are different. The results in this chapter can also be found in [27, 31].

7.1 The three probabilistic relations

The three game variants differ from each other in their definition of q_{ij}. For two (n, σ) partitions $\pi_i = (i_1, \ldots, i_n)$, $\pi_j = (j_1, \ldots, j_n)$,

(i) the first game variant defines q_{ij} as

$$q_{ij}^{\mathbf{P}} = \frac{\#\{(k,l) \mid i_k > j_l\}}{n^2} + \frac{\#\{(k,l) \mid i_k = j_l\}}{2n^2}\,, \tag{7.1}$$

(ii) the second game variant defines q_{ij} as

$$q_{ij}^{\mathbf{M}} = \frac{\#\{k \mid i_k > j_k\}}{n} + \frac{\#\{k \mid i_k = j_k\}}{2n}\,, \tag{7.2}$$

(iii) and the third game variant defines q_{ij} as

$$q_{ij}^{\mathbf{L}} = \frac{\#\{k \mid i_k > j_{n-k+1}\}}{n} + \frac{\#\{k \mid i_k = j_{n-k+1}\}}{2n}\,. \tag{7.3}$$

The first (second, third) game variant is denoted as an $(n, \sigma)_{\mathbf{P}}$ game ($(n, \sigma)_{\mathbf{M}}$ game, $(n, \sigma)_{\mathbf{L}}$ game). Here, **P** refers to the product copula, **M** to the minimum copula and **L** to the Łukasiewicz copula, which are the respective copulas used for the coupling of the random variables [64], see also the previous chapter (representations (6.7) and (6.9)). Note that the $(n, \sigma)_{\mathbf{P}}$ game is precisely the (n, σ) game that was considered in Chapter 5. As one can see from Figure 7.1 it is no coincidence that random variables coupled by the $T_{\mathbf{M}}$-copula ($T_{\mathbf{L}}$-copula) are often called comonotonic (countermonotonic).

Consider e.g. the $(4, 16)$ partitions $\pi_1 = (1, 2, 5, 8)$ and $\pi_2 = (2, 3, 5, 6)$. Figure 7.1 shows graphically, for each considered game variant, which parts of the partitions have to be compared. We obtain $q_{12}^{\mathbf{P}} = (0 + 0.5 + 2.5 + 4)/16 = 7/16$, $q_{12}^{\mathbf{M}} = 0 + 0 + 1/8 + 1/4 = 3/8$ and $q_{12}^{\mathbf{L}} = 0 + 0 + 1/4 + 1/4 = 1/2$. Clearly, in strict sense the dice metaphor only applies to the case of the $T_{\mathbf{P}}$-copula. Indeed, for the extreme copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ the pairwise comparison of dice reduces to the pairwise comparison of ordered lists, in the sense that for two such ordered lists, each element of the first list is compared to exactly one element of the second list, where the order of the elements determines which elements are compared to each other. This is why we drop the use of the term dice in this chapter, as we only consider the last two game types. When it is better suited to explicitly mention the partitions defining q_{ij}, we will use the notation Q_{π_i,π_j}, which is the same notation as was used in Chapter 5.

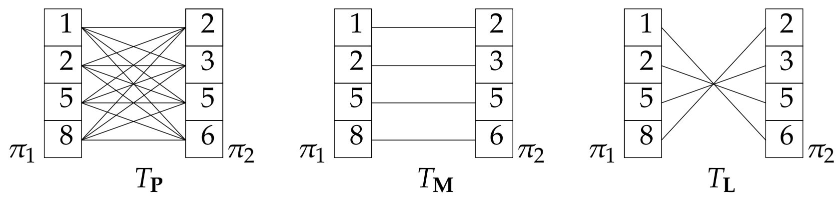


Figure 7.1: *The three game types for a specific example.*

In the next two sections, the optimal strategies of the $(n, \sigma)_{\mathbf{M}}$ games and the $(n, \sigma)_{\mathbf{L}}$ games are obtained. Both sections start with a subsection that bundles the results, after which a subsection follows in which these results are proven.

7.2 Optimal strategies for $(n, \sigma)_{\mathbf{M}}$ games

7.2.1 Results

The following lemma states a remarkable result about the integers occurring as parts of an optimal strategy in an $(n, \sigma)_{\mathbf{M}}$ game.

LEMMA – 7.2.1: *The only optimal strategy in an $(n, \sigma)_{\mathbf{M}}$ game, with $n \geq 3$, for which the highest part is strictly greater than 5 is $(2, 4, 6)$, a strategy of the $(3, 12)_{\mathbf{M}}$ game.*

The above lemma will be crucial in our proof of the following theorem.

THEOREM – 7.2.2: *An $(n, \sigma)_{\mathbf{M}}$ game has optimal strategies if and only if one of the following three mutually exclusive conditions is satisfied.*

(i) $n \leq 2$

(ii) $(n, \sigma) = (3, 12)$

(iii) $n > 2$ *and there exist* $t_1, \ldots, t_5 \in \mathbb{N}$ *such that they are a solution of the following system:*

$$\begin{cases} t_1 + t_2 + t_3 + t_4 + t_5 = n \\ t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = \sigma \\ t_3 > 0 \Rightarrow t_2 + 2 > (t_3 - 1) + t_4 + t_5 \\ t_4 > 0 \Rightarrow t_3 + 2 > t_1 + (t_4 - 1) + t_5 \\ t_5 > 0 \Rightarrow t_4 + 2 > t_1 + t_2 + (t_5 - 1) \end{cases} \tag{7.4}$$

We are also able to describe the optimal strategies of the $(n, \sigma)_{\mathbf{M}}$ games. We first handle some special cases.

PROPOSITION – 7.2.3:

1. *The* $(1, \sigma)_{\mathbf{M}}$ *game: the unique strategy* (σ) *is optimal.*
2. *The* $(2, \sigma)_{\mathbf{M}}$ *game: all* $\lfloor \frac{\sigma}{2} \rfloor$ *strategies are optimal.*
3. *The* $(3, 12)_{\mathbf{M}}$ *game:* $(2, 4, 6)$ *is the only optimal strategy.*

All other optimal strategies are identified in the next proposition.

PROPOSITION – 7.2.4: *All optimal strategies of* $(n, \sigma)_{\mathbf{M}}$ *games that are not covered by Proposition 7.2.3 have multiplicity representation* $(1^{t_1}2^{t_2}3^{t_3}4^{t_4}5^{t_5})$*, where* $(t_1, \ldots, t_5)$ *is a solution of* (7.4).

However, a closed formula expressing the number of optimal strategies of an arbitrary $(n, \sigma)_{\mathbf{M}}$ game has not yet been found.

EXAMPLE – 7.2.5:
The $(5, 16)_{\mathbf{M}}$ game has 37 strategies and only one optimal strategy, namely $\pi = (2, 2, 3, 4, 5)$ for which $(t_1, t_2, t_3, t_4, t_5) = (0, 2, 1, 1, 1)$. One can easily verify that conditions (7.4) are satisfied for π. Moreover, none of the other $(5, 16)$ partitions satisfy these conditions. ◀

7.2.2 Proof

We start this subsection by introducing increment and decrement operations, which will be essential in the subsequent proof. The following definitions were already introduced in Chapter 5, however they are repeated here for clarity. Any (n, σ) partition π_2 can be constructed starting from any (n, σ) partition π_1 using increment/decrement operations. An increment/decrement operation on an (n, σ) partition is an operation in which one part of the partition is increased by 1 (the increment operation) while a second part is decreased by 1 (the decrement operation), resulting in another (n, σ) partition. In the case of the $(n, \sigma)_{\mathbf{M}}$ game, we represent an (n, σ) partition as a nondecreasingly ordered column of integers and we apply an increment or decrement operation to a specific row. Consider e.g. the $(5, 12)$ partitions $\pi_1 = (1, 1, 3, 3, 4)$ and $\pi_2 = (1, 1, 2, 3, 5)$ (for which $q_{12}^{\mathbf{M}} = 1/2$):

$$
\begin{array}{c|c}
\pi_1 & \pi_1 \\
\hline
1 & 1 \\
1 & 1 \\
3 & 3 \\
3 & 3 \\
4 & 4
\end{array}
\quad \rightarrow \quad
\begin{array}{c|c}
\pi_1 & \pi_2 \\
\hline
1 & 1 \\
1 & 1 \\
3 & \lfloor 2 \rfloor \\
3 & 3 \\
4 & \lceil 5 \rceil
\end{array}
$$

We see that the increment operation is applied to row 5 and the decrement operation to row 3. For brevity, we say that row 5 is incremented and row 3 is decremented. In the present case, row 4 cannot be decremented instead of row 3, since the then obtained column of integers would no longer be nondecreasing. Through a concatenation of these increment/decrement operations, any (n, σ) partition π_2 can be obtained from the partition π_1. We can restrict these concatenations in the sense that once a row has been incremented (resp. decremented), it cannot be decremented (resp. incremented). Indeed, an increment operation followed later by a decrement operation (and vice versa) applied to the same row cancel each other out and can therefore be ignored. A concatenation of increment/decrement operations transforming π_1 into π_2 will be called a (π_1, π_2) transformation.

Let ν_i (resp. ν_d) denote the number of different incremented (resp. decremented) rows in the (π_1, π_2) transformation. We then have that $Q_{\pi_1,\pi_2} > 1/2 \Leftrightarrow \nu_i < \nu_d$. This is easily seen by noting that ν_i (resp. ν_d) is nothing else but $\#\{j \mid i_j < i'_j\}$ (resp. $\#\{j \mid i_j > i'_j\}$). In general it thus holds that

$$Q_{\pi_1,\pi_2} = \frac{1}{2} - \frac{\nu_i - \nu_d}{2\,n}\,. \tag{7.5}$$

We illustrate (7.5) on some more examples.

EXAMPLE – 7.2.6:

(i) Consider the $(5, 18)$ partitions $\pi_1 = (2, 3, 4, 4, 5)$ and $\pi_2 = (1, 3, 3, 5, 6)$. The transformation of π_1 in π_2 goes (e.g.) as follows:

$$\begin{array}{c|c} \pi_1 & \pi_1 \\ \hline 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 4 & 4 \\ 5 & 5 \end{array} \;\longrightarrow\; \begin{array}{c|c} \pi_1 & \pi_1^{(1)} \\ \hline 2 & \lfloor 1 \rfloor \\ 3 & 3 \\ 4 & 4 \\ 4 & \lceil 5 \rceil \\ 5 & 5 \end{array} \;\longrightarrow\; \begin{array}{c|c} \pi_1 & \pi_2 \\ \hline 2 & 1 \\ 3 & 3 \\ 4 & \lfloor 3 \rfloor \\ 4 & 5 \\ 5 & \lceil 6 \rceil \end{array}$$

We obtain $\nu_i = \nu_d = 2$ and therefore $Q_{\pi_1,\pi_2} = 1/2$.

(ii) Consider the $(5, 18)$ partitions $\pi_1 = (2, 3, 4, 4, 5)$ and $\pi_2 = (1, 1, 5, 5, 6)$. The transformation of π_1 in π_2 now goes (e.g.) as follows:

$$\begin{array}{c|c} \pi_1 & \pi_1 \\ \hline 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 4 & 4 \\ 5 & 5 \end{array} \;\longrightarrow\; \begin{array}{c|c} \pi_1 & \pi_1^{(1)} \\ \hline 2 & \lfloor 1 \rfloor \\ 3 & 3 \\ 4 & 4 \\ 4 & \lceil 5 \rceil \\ 5 & 5 \end{array} \;\longrightarrow\; \begin{array}{c|c} \pi_1 & \pi_1^{(2)} \\ \hline 2 & 1 \\ 3 & \lfloor 2 \rfloor \\ 4 & 4 \\ 4 & 5 \\ 5 & \lceil 6 \rceil \end{array} \;\longrightarrow\; \begin{array}{c|c} \pi_1 & \pi_2 \\ \hline 2 & 1 \\ 3 & \lfloor\lfloor 1 \rfloor\rfloor \\ 4 & \lceil 5 \rceil \\ 4 & 5 \\ 5 & 6 \end{array}$$

Here, we obtain $\nu_i = 3$ and $\nu_d = 2$, which implies $Q_{\pi_1,\pi_2} = 1/2 - 1/(10)$. ◀

The above reasoning will be applied below. We discuss all (n, σ) partitions by considering the next three cases. After these cases have been considered, we then conclude by determining the maximum value for the parts of an optimal strategy in an $(n, \sigma)_{\mathbf{M}}$ game.

Case 1: $n \leq 2$.

When $n = 1$ there is only one (n, σ) partition, when $n = 2$ it is obvious that all (n, σ) partitions play a draw. Indeed, for two $(2, \sigma)$ partitions $\pi_1 = (a_1, \sigma - a_1)$ and $\pi_2 = (b_1, \sigma - b_1)$, $a_1 \leq b_1$, it holds that either $\sigma - a_1 > \sigma - b_1$ when $a_1 < b_1$, or $\sigma - a_1 = \sigma - b_1$ when $a_1 = b_1$. The first two parts of Proposition 7.2.3 and (i) of Theorem 7.2.2 are therefore already proven. ■

Case 2: Partitions satisfying

$$(\exists\, j > 1)\ (t_{j+1} > 0 \wedge n \geq 2\, t_j + 3 + t_{j-1})\,. \tag{7.6}$$

These partitions are not optimal. Indeed, construct π_2 starting from π_1 by decrementing all t_j parts having value j by 1, decrementing a part having value $j + 1$ by two and incrementing $t_j + 2$ other parts from π_1, all different from $j - 1$. This transformation can be done using increment/decrement operations. The idea behind the transformation is that there will be two decrement operations applied to the row on which the first occurrence of $j + 1$ is situated in the original partition π_1, while all increment operations are applied to different rows. Using (7.5) we obtain that $Q_{\pi_1,\pi_2} = (n-1)/(2n)$ and π_1 is therefore not optimal. Essential for this construction is that (7.6) holds, as this condition must be satisfied in order to be able to do all the increment operations on different rows. ■

Example – 7.2.7:

Consider the $(8, 23)$ partition $\pi_1 = (1, 2, 2, 3, 3, 3, 4, 5)$. Condition (7.6) is satisfied for $j = 4$. If we choose $\pi_2 = (2, 3, 3, 3, 3, 3, 3, 3)$, we obtain $Q_{\pi_1,\pi_2} = (n-1)/(2n) = 7/(16) < 1/2$.

π_1	π_1		π_1	$\pi_1^{(1)}$		π_1	$\pi_1^{(2)}$		π_1	π_2
1	1		1	1		1	1		1	2
2	2		2	2		2	3		2	3
2	2		2	3		2	3		2	3
3	3	$\rightarrow$	3	3	$\rightarrow$	3	3	$\rightarrow$	3	3
3	3		3	3		3	3		3	3
3	3		3	3		3	3		3	3
4	4		4	3		4	3		4	3
5	5		5	5		5	4		5	3

In the last transformation, we see that the decremented part is again on the row where the first occurrence of $j + 1$ is situated in π_1, which is the reason why $Q_{\pi_1,\pi_2} < 1/2$. ◀

Case 3: All partitions not yet covered above are optimal.

These partitions satisfy

$$n \geq 3 \wedge (\forall\, j > 1)\ (t_{j+1} > 0 \Rightarrow n < 2\, t_j + 3 + t_{j-1})\,. \tag{7.7}$$

Before presenting the proof, we fix some notation. We say that an increment or decrement operation yields a decrementable (resp. incrementable) row, if after the increment or decrement operation a row becomes available for a decrement (resp. increment) operation and that row was not available before the increment or decrement operation was performed. Consider e.g. $\pi_1 = (2,3,3,5)$. It holds that incrementing row 3 yields an incrementable row (namely row 2) while incrementing row 4 does not yield an incrementable or decrementable row. Indeed, by incrementing row 3 we obtain $\pi_1' = (2,3,4,5)$ and in this partition row 2 is incrementable while it was not incrementable in partition π_1. Incrementing row 4 yields $\pi_1'' = (2,3,3,6)$ and all incrementable or decrementable rows are the same for π_1 and π_1''. We will also use the notions of first increment (resp. decrement) operation on a row and first increment (resp. decrement) operation on the same row. The former denotes an increment (resp. decrement) operation done on a row that has not yet been incremented or decremented in the process of transforming π_1 into π_2. The latter denotes the increment (resp. decrement) operation in the transformation step in which it happens for the first time that a row is incremented (resp. decremented) for a second time.

We now prove that all (n,σ) partitions π_1 satisfying (7.7) are optimal strategies. Suppose that there exists an (n,σ) partition π_2 that wins from π_1. Partition π_2 can again be obtained from partition π_1 using increment/decrement operations. From (7.5) we know that the number of incremented rows must be higher than the number of decremented rows. We now show that this implies that (7.6) holds, which contradicts (7.7).

Notice first that if (7.6) would be satisfied then there exists an (n,σ) partition π_2 that wins from π_1 such that there exists a (π_1,π_2) transformation in which the first decrement on the same row happens earlier than the first (if any) increment operation on the same row. Conversely, when there exists a (π_1,π_2) transformation such that the first decrement on the same row happens earlier than the first (if any) increment on the same row, then (7.6) must hold.

Since we suppose that π_1 is not optimal, the only case in which (7.6) would not be satisfied is when for all (n,σ) partitions π_2 that win from π_1, all possible (π_1,π_2) transformations would be such that the first increment on the same row happens earlier or at the same time as the first decrement on the same row. We therefore only need to show that a first increment on the same row is useless for obtaining rows that can be decremented and also for obtaining rows that can be incremented for the first time. In the next paragraph, we will show that an increment on the same row can only yield another row that has already been incremented. As the number of incremented rows must be higher than the number of decremented rows, it is therefore never necessary for the first increment on the same row to happen earlier or at the same time as the first decrement on the same row.

It is obvious that, in general, not all rows can be used for an increment. For example, for the partition $\pi = (3,3,3)$ only the third row can be incremented. However, a first increment on a row can yield a row that can be incremented for the first time. For example, decrementing the first row and incrementing the last row of π results in $\pi' = (2,3,4)$. The increment of row 3 makes it possible to use row 2 of π' for an increment operation. This was impossible for partition π. A second increment on the same row, however, never yields a row that can be incremented for the first time. It is also obvious that an increment operation never yields a decrementable row.

As $Q_{\pi_1,\pi_2} < 1/2$, the above reasoning shows that there always exists a (π_1,π_2) transformation in which the second decrement on a certain row happens before the second increment (if any) on some other row. But this is impossible, since (7.7) would then not be satisfied. ∎

As Case 2 proved that all (n,σ) partitions, with $n \geq 3$, not satisfying (7.7) are not optimal, Step 3 proves that an (n,σ) partition, with $n \geq 3$, is optimal if and only if (7.7) is satisfied. ∎

We now determine a maximum value μ for the parts of an optimal strategy π_1 in any $(n,\sigma)_{\mathbf{M}}$ game with $n \geq 3$.

First of all, it can be easily verified that $\pi_1 = (2,4,6)$ is the only optimal strategy in the $(3,12)_{\mathbf{M}}$ game, which implies $\mu > 5$.

Note that when an integer $j > 1$ exists such that $t_{j-1} = t_j = 0$ and $t_{j+1} \neq 0$, it holds that (7.7) is not satisfied for this value j and the partition therefore is not an optimal strategy. We can therefore assume $t_{j-1} = t_j = 0 \Rightarrow t_{j+1} = 0$, for any $j > 1$. This implies that $n \geq \lceil \mu/2 \rceil$ and that there are at least $\lceil \mu/2 \rceil \geq 3$ distinct parts in π_1. When $t_i \neq 0$ for all $2 \leq i \leq 6$, one can verify that (7.7) is not satisfied. Suppose therefore for some $1 < i < 6$ that $t_i = 0$ and $t_{i+1} \neq 0$, then it must hold that $n < 3 + t_{i-1}$. As there are at least 3 distinct parts, it holds that $n - t_{i-1} \geq 2$. This in turn implies that $n = 2 + t_{i-1}$, which implies that there are exactly three distinct numbers in the partition, implying $\mu \leq 6$. When $\mu = 6$, the fact that there are exactly three distinct numbers implies that $t_{2i-1} = 0$ and $t_{2i} > 0$, for $i \in \{1,2,3\}$. As $n < 3 + t_2$, $n < 3 + t_4$ and $n = t_2 + t_4 + t_6$ we obtain $t_2 = t_4 = t_6 = 1$, resulting in $\pi_1 = (2,4,6)$. Note that π_1 clearly satisfies (7.7).

The above reasoning proves Lemma 7.2.1 and also the third part of Proposition 7.2.3. ∎

The above results can now be combined to prove Proposition 7.2.4. Indeed, from the above reasoning it follows that an (n,σ) partition π_1 is optimal if and only if either $n < 3$, or (7.7) holds. If $n \geq 3$ and $\pi_1 \neq (2,4,6)$, then we also know from the above reasoning that the optimal strategy contains no parts strictly greater than 5 and therefore has as multiplicity representation $(1^{t_1}2^{t_2}3^{t_3}4^{t_4}5^{t_5})$. We can now conclude the proof of Proposition 7.2.4 by making the following remarks. Firstly, it is obvious that

$$t_1 + t_2 + t_3 + t_4 + t_5 = n\,, \quad t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = \sigma\,, \tag{7.8}$$

is equivalent to saying that the partition is an (n,σ) partition and that it contains no parts strictly greater than 5. Secondly, the three other conditions

$$\begin{cases} t_3 > 0 \Rightarrow t_2 + 2 > (t_3 - 1) + t_4 + t_5 \\ t_4 > 0 \Rightarrow t_3 + 2 > t_1 + (t_4 - 1) + t_5 \\ t_5 > 0 \Rightarrow t_4 + 2 > t_1 + t_2 + (t_5 - 1) \end{cases}$$

are merely a restatement of (7.7) using (7.8). This proves Proposition 7.2.4. Theorem 7.2.2 now follows immediately. ∎

7.3 Optimal strategies for $(n, \sigma)_{\mathbf{L}}$ games

7.3.1 Results

While not all $(n, \sigma)_{\mathbf{P}}$ or $(n, \sigma)_{\mathbf{M}}$ games have an optimal strategy, the situation is different for $(n, \sigma)_{\mathbf{L}}$ games.

THEOREM – 7.3.1: *All $(n, \sigma)_{\mathbf{L}}$ games have at least one optimal strategy.*

The exact characterization of these optimal strategies in an $(n, \sigma)_{\mathbf{L}}$ game is given by the following proposition.

PROPOSITION – 7.3.2: *Consider an (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$ and let*

$$a = \lfloor \frac{n}{2} \rfloor + 1 \,, b = \lfloor \frac{\sigma - n}{a} \rfloor + 1 \,, c = \begin{cases} n + 1 - \lfloor \frac{\sigma - n}{b - 1} \rfloor & , \textit{when } b \neq 1 \,, \\ n + 1 - (\sigma - n) & , \textit{when } b = 1 \,. \end{cases} \tag{7.9}$$

The (n, σ) partition π is an optimal strategy of an $(n, \sigma)_{\mathbf{L}}$ game if and only if one of the following four mutually exclusive conditions holds:

(i) $\sigma - n \leq \lfloor n/2 \rfloor$ *and:*

- $\pi = (1^{c-1} 2^{n-c+1})$.

(ii) $(n, \sigma) = (n, 2n)$, $n \geq 1$ *and:*

- $\pi = (1^m 2^{(n-2m)} 3^m)$, $m \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

(iii) $(n, \sigma) = (2l, \sigma)$, $l > 0$, $\sigma \neq 2n$, $\sigma > 3l$ *and:*

- $(i_c = b \ \wedge \ \sigma \neq l(b+2) + b - 1)$, *or*
- $i_{l+1} \geq b + 1$, *or*
- $\pi = (1^{l-1} b^2 (b+1)^{l-1})$, *implying* $(n, \sigma) = (2l, l(b+2) + b - 2)$.

(iv) $(n, \sigma) = (2l+1, \sigma)$, $l \geq 0$, $\sigma \neq 2n$, $\sigma > 3l + 1$ *and:*

- $i_c = b$, *or*
- $\pi = (1^l b^1 (b+1)^l)$, *implying* $(n, \sigma) = (2l+1, l(b+2) + b)$.

EXAMPLE – 7.3.3:

(i) The $(6, 17)_{\mathbf{L}}$ game has 44 strategies of which 5 are optimal: $(1^3 4^2 6^1)$, $(1^3 4^1 5^2)$, $(1^2 2^1 4^2 5^1)$, $(1^2 3^1 4^3)$ and $(1^1 2^2 4^3)$. Note that for this game $b = 3$ and it therefore holds that $\sigma = l(b+2) + b - 1$ (with $l = 3$), which implies that only the partitions for which $i_{l+1} \geq b + 1$ are optimal strategies.

(ii) The $(8,23)_{\mathbf{L}}$ game has 146 strategies and only one optimal strategy, given by $(1^3 4^5)$. Indeed, $b=c=4$ and there are no strategies satisfying $i_{l+1} = b+1$.

(iii) The $(9,23)_{\mathbf{L}}$ game has 123 strategies of which two are optimal: $(1^2 3^7)$ and $(1^4 3^1 4^4)$. As $b=c=3$, the first partition corresponds to the case $i_c = b$ while the second one is of the form $(1^l b^1 (b+1)^l)$. ◀

We can also state the number of optimal strategies in function of $p(N,M,n)$, which was introduced in Definition 1.4.6.

Proposition – 7.3.4: *Let $p_n(M,N) = \sum_{i=0}^{N} p(N,M,N-i)p(N,n-M,i)$ and let*

$$\Sigma_1 = \sigma - n - \lfloor \frac{\sigma - n}{b-1} \rfloor (b-1)\,, \quad \Sigma_2 = \sigma - l(b+2)\,, \tag{7.10}$$

with b and c defined in (7.9). The number of optimal strategies in an $(n,\sigma)_{\mathbf{L}}$ game, here denoted as $\nu(n,\sigma)$, is then given in one of the following 5 mutually exclusive cases ($l>0$).

(i) $\sigma - n \leq \lfloor n/2 \rfloor \vee n = 1$:
$\nu(n,\sigma) = 1$.

(ii) $(n,\sigma) = (n,2n) \wedge n > 1$:
$\nu(n,\sigma) = \lfloor \frac{n}{2} \rfloor + 1$.

(iii) $(n,\sigma) = (2l,\sigma) \wedge \sigma = l(b+2)+b-1 \wedge \sigma > 3l$:
$\nu(n,\sigma) = p_n(l,\Sigma_2)$.

(iv) $(n,\sigma) = (2l,\sigma) \wedge \sigma \neq 2n \wedge l(b+2)+b-1 > \sigma > 3l$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1) + p_n(l,\Sigma_2) + \lceil \frac{l-c}{l+1-c} \rceil \lfloor \frac{\sigma}{l(b+2)+b-2} \rfloor$.

(v) $(n,\sigma) = (2l+1,\sigma) \wedge \sigma \neq 2n \wedge \sigma > 3l+1$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1) + \lceil \frac{l+1-c}{l+2-c} \rceil \lfloor \frac{\sigma}{l(b+2)+b} \rfloor$.

7.3.2 Proof

In the next five cases Theorem 7.3.1 and Proposition 7.3.2 are proven and thereafter, using the results obtained in these cases, Proposition 7.3.4 is proven. Throughout this subsection, when using an (n,σ) partition π_1 (resp. π_2), its parts will be written as i_j (resp i'_j), for $j \in \mathbb{N}[1,n]$.

Case 1: $\sigma - n \leq \lfloor n/2 \rfloor$.

We start by considering the special case of $(n,\sigma)_{\mathbf{L}}$ games for which $\sigma - n \leq \lfloor n/2 \rfloor$, corresponding to (i) of Proposition 7.3.2. Note that this condition is equivalent to $b=1$. It is obvious that $\pi_1 = (1^{c-1} 2^{n-c+1})$, with $c = n+1-(\sigma - n)$, is an (n,σ) partition and that π_1 wins from any other (n,σ) partition π_2. Indeed, let $k = \#\{j \mid i'_j = 1\}$ and $m = \#\{j \mid i'_j > 1\} = n-k$. As $\pi_2 \neq \pi_1$ it holds that $k \geq c > \lceil n/2 \rceil$ and $Q_{\pi_2,\pi_1} = (2m + (c-m-1))/(2n) = (n-k+c-1)/(2n) < 1/2$. ∎

In the remainder, we will assume $\sigma - n > \lfloor n/2 \rfloor$ and therefore $b > 1$. A partition will be represented graphically as a Ferrers graph, which was introduced in Section 1.4. Although this representation is not essential in the proof, it helps to visualize the meaning of some variables that will be used.

For an (n, σ) partition π_1 we utilize the following values, which were already introduced in Proposition 7.3.2 (recall that $b > 1$):

$$a = \lfloor \frac{n}{2} \rfloor + 1, \quad b = \lfloor \frac{\sigma - n}{a} \rfloor + 1, \quad c = n + 1 - \lfloor \frac{\sigma - n}{b-1} \rfloor . \tag{7.11}$$

In words, b denotes the highest possible value for $i_{\lceil n/2 \rceil}$ and c denotes the lowest possible value j such that $i_j = b$ is possible. Therefore, an (n, σ) partition π_1 for which $i_{\lceil n/2 \rceil} = b$ surely exists.

Case 2: $n = 2l + 1 \wedge i_{l+1} < b$, or, $n = 2l \wedge i_l < b \wedge i_{l+1} < b + 1$.

When $n = 2l + 1$, any (n, σ) partition π_1 for which $i_{l+1} < b$ loses from any partition π_2 for which $i'_{l+1} = b$ and is therefore not an optimal strategy. Indeed, it then holds that $i'_{n-j} > i_{j+1}$, for any $0 \leq j \leq l$, which implies $Q_{\pi_2,\pi_1} \geq (l+1)/n > 1/2$. When $n = 2l$, then any partition π_1 for which $i_l < b$ and $i_{l+1} < b + 1$ loses from any partition π_2 for which $i'_l = b$. Indeed, it then holds that $i'_{n-j} > i_{j+1}$, for any $0 \leq j < l$ and $i'_l \geq i_{l+1}$, which again implies $Q_{\pi_2,\pi_1} > 1/2$. We can therefore already exclude these partitions π_1 as they are not optimal strategies. Note that this does not exclude a priori the possibility for an $(n, \sigma)_{\mathbf{L}}$ game to have optimal strategies. ■

Case 3: $(n, \sigma) = (n, 2n)$.

In this case, all optimal strategies π_1 are given by

$$\pi_1 = (1^m 2^{n-2m} 3^m), m \in \{0, 1, \ldots, \lfloor n/2 \rfloor\} . \tag{7.12}$$

We first prove that the strategies of type (7.12) are optimal. Let π_2 be another $(n, 2n)$ partition, with $k' = \#\{j \mid i'_j = 1\}$ and $m' = \#\{j \mid i'_j > 2\}$. As $\sigma = 2n$, we have that $k' \geq m'$. When $m' \leq m$, we obtain $Q_{\pi_2,\pi_1} \leq (2m + (n - 2m))/(2n) = 1/2$. When $m' > m$, we obtain

$$Q_{\pi_2,\pi_1} \leq (2m' + n - k' - m')/(2n) \leq 1/2 . \tag{7.13}$$

We now prove that the strategies (7.12) are the only optimal strategies in the $(n, 2n)_{\mathbf{L}}$ game. For any π_2, with k' and m' as defined above, such that $k' > m' > 0$, it follows from (7.13) that $Q_{\pi_2,\pi_1} < 1/2$. When $m' = 0$ it holds that $\pi_2 = (2^n)$, which is of type (7.12). When $k' = m'$ partition π_2 is of type (7.12). This proves (ii) of Proposition 7.3.2. ■

We now subdivide the not yet covered $(n, \sigma)_{\mathbf{L}}$ games into those where n is even and those where n is odd.

Case 4: $n = 2l \wedge \sigma \neq 2n \wedge b \neq 1$.

Subcase 4.1: $(2l, \sigma)$ partitions π_1 such that $i_{l+1} \geq b + 1$.

Before subdividing the $(2l, \sigma)$ games further, we consider these partitions π_1, which can occur in any of the still to be considered $(2l, \sigma)$ games. All $(2l, \sigma)$ partitions π_1 for which $i_{l+1} \geq b + 1$, if there are any, are optimal strategies.

Indeed, suppose such a partition π_1 exists. For any $(2l, \sigma)$ partition π_2 it holds that $i'_l < b+1$. Therefore, $i_{n-j} > i'_{j+1}$, for any $0 \leq j < l$, which implies $Q_{\pi_1,\pi_2} \geq 1/2$. This corresponds to the second part of (iii) of Proposition 7.3.2. ■

Subcase 4.2: (n, σ) games satisfying

$$\sigma = l(b+2) + b - 1\,. \tag{7.14}$$

At least one (n, σ) partition π_1 satisfying $i_{l+1} \geq b+1$ exists and these (n, σ) partitions comprise all optimal strategies. Indeed, any partition π_2 for which $i'_{l+1} < b+1$ loses from the partition $\pi_1 = (1^{l-1} b^1 (b+1)^l)$. This explains the condition $\sigma \neq l(b+2) + b - 1$ in the first part of (iii) of Proposition 7.3.2. ■

EXAMPLE – 7.3.5:
Consider the $(8, 27)_{\mathbf{L}}$ game, which has 352 strategies of which 10 are optimal, and for which (7.14) clearly holds ($b = \lfloor 19/5 \rfloor + 1 = 4$). The Ferrers graph (with some annotations) for the partition $(1^3 b^1 (b+1)^4)$ is shown in Figure 7.2.

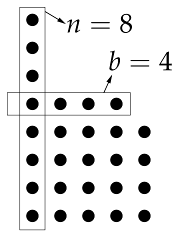


Figure 7.2: *Ferrers graph for the* $(8, 27)$ *partition* $(1, 1, 1, 4, 5, 5, 5, 5)$.

◀

We now investigate the last remaining class of $(2l, \sigma)_{\mathbf{L}}$ games.
Subcase 4.3: (n, σ) games for which $\sigma \neq l(b+2) + b - 1$.

We only need to consider (n, σ) partitions for which $i_l = i_{l+1} = b$ and $b \neq 1$, as the other types of (n, σ) partitions were already encountered before. The fact that (7.14) is not satisfied implies that for any two partitions π_1 satisfying $i_l = b$ and π_2 satisfying $i'_{l+1} \geq b+1$ it holds that $Q_{\pi_1,\pi_2} = 1/2$. It therefore suffices to investigate Q_{π_1,π_2} with π_1 satisfying $i_l = i_{l+1} = b$ and π_2 satisfying $i'_l = i'_{l+1} = b$. Note that the strict inequality

$$\sigma < l(b+2) + b - 1 \tag{7.15}$$

must then hold. Indeed, $\sigma > l(b+2) + b - 1$ implies that $b < (\sigma - n + 1)/(l + 1) \leq \lfloor (\sigma - n + l + 1)/(l+1) \rfloor = b$, which is of course impossible.

We first introduce a useful lemma, considering both n even and n odd, which will make the subsequent proof and the proof of Case 5 simple.

LEMMA – 7.3.6: *For an* (n, σ) *partition* π_1 *(n even or odd) with* $i_{\lceil n/2 \rceil} = i_{\lceil n/2 \rceil + 1} = b > 1$ *(b defined by (7.11)), let* $s = \min\{j \mid i_j = b\}$ *and* $t = \max\{j \mid i_j = b\}$. *It then holds that* $t \geq n + 2 - s$ *if* $s < \lceil n/2 \rceil \wedge b > 2$, *and* $t \geq n + 1 - s$ *if* $s = \lceil n/2 \rceil \vee b = 2$.

Proof:
Let $n = 2l$ (resp. $n = 2l + 1$) when n is even (resp. odd). It holds that $c \leq s \leq \lceil n/2 \rceil$ (c defined by (7.11)) and $t \geq \lceil n/2 \rceil + 1$. By definition of s and t it must hold that

$$\sigma \geq s - 1 + (t - s + 1)b + (n - t)(b + 1)\,, \tag{7.16}$$

or equivalently,

$$t \geq n - \sigma + nb - (s - 1)(b - 1)\,. \tag{7.17}$$

First assume n is even. As (7.15) holds, we obtain (adding s to both sides of (7.17))

$$t + s > n - l(b + 2) - b + 1 + nb - (s - 1)(b - 1) + s\,,$$

which simplifies to

$$t + s > n + (l - s)(b - 2)\,, \tag{7.18}$$

from which the desired inequalities immediately follow.

Now assume n is odd. From the tautology $b - 1 < b$, it follows that $\lfloor (\sigma - n)/(l + 1) \rfloor < b$, which implies $\sigma - n < (l + 1)b$, finally implying $\sigma < l(b + 2) + b + 1$. Together with (7.17) this implies that

$$t + s > n + (l + 1 - s)(b - 2)\,, \tag{7.19}$$

from which the desired inequalities again follow. ∎

We now continue with Subcase 4.3. Suppose $i'_c = i'_{l+1} = i_l = i_{l+1} = b$, with $n = 2l$. Let $r = \max\{j \mid i'_j = b\}$ and let s and t be defined as in the above lemma. Hence, the parts of π_1 and π_2 satisfy

$$\begin{cases} i_j < b, \text{ if } 1 \leq j < s\,, \\ i_j = b, \text{ if } s \leq j \leq t\,, \\ i_j > b, \text{ if } t < j \leq n\,, \end{cases} \qquad \begin{cases} i'_j < b, \text{ if } 1 \leq j < c\,, \\ i'_j = b, \text{ if } c \leq j \leq r\,, \\ i'_j > b, \text{ if } r < j \leq n\,. \end{cases}$$

It now holds that

$$\begin{aligned} Q_{\pi_2,\pi_1} &= \frac{1}{n}\Bigg(\max(s-1, n-r) + \\ &\qquad \frac{1}{2}\big(\min(n+1-c, t) - \max(s-1, n-r)\big)\Bigg) \\ &= \frac{1}{2n}\big(\min(n+1-c, t) + \max(s-1, n-r)\big)\,. \end{aligned}$$

First consider $s = c$. Using Lemma 7.3.6, we then obtain $Q_{\pi_2,\pi_1} = (n + 1 - c + s - 1)/(2n) = 1/2$. Partitions π_1 and π_2 satisfying $i_c = i_{l+1} = b$ resp. $i'_c = i'_{l+1} = b$ therefore play a draw. Next, consider $s > c$, implying

$$Q_{\pi_2,\pi_1} = \frac{1}{2n}\big(\min(n+1-c, t) + s - 1\big)\,.$$

If $t \geq n+1-c$, then $Q_{\pi_2,\pi_1} = (n+s-c)/(2n)$, which implies $Q_{\pi_2,\pi_1} > 1/2$ for $s \neq c$. If $t < n+1-c$ then it holds that $Q_{\pi_2,\pi_1} = (t+s-1)/(2n) \geq 1/2$ (again using Lemma 7.3.6). Moreover, when $s < l$ or $b > 2$, the same lemma implies $Q_{\pi_2,\pi_1} > 1/2$. The above already proves the first part of *(iii)* of Proposition 7.3.2.

Partitions satisfying $s = l$ and the case $b = 2$ need more investigation, to see if there are other optimal strategies possible. We therefore investigate when it holds that $Q_{\pi_2,\pi_1} = 1/2$, or equivalently when $t+s-1 = n$. Inequality (7.16) is then equivalent to

$$\sigma \geq 2n + t(b-2)\,. \tag{7.20}$$

The definition of b from (7.11) implies $b(l+1) > \sigma - n$ and combining this with (7.20), we obtain the strict inequality

$$(b-2)(t-l) < b\,. \tag{7.21}$$

Inequality (7.21) is only satisfied when $b = 2$ or when $t = l+1$ (recall that $b = 1$ is excluded and that $t \geq l+1$). Indeed, when $t - l > 1$, it holds that (7.21) is equivalent to $b < 2 + 2/(t-l-1)$, which can only hold when $b = 2$. When $b = 2$ it holds that $\sigma \geq 2n$ and the definition of b then implies $\sigma = 2n$ or $\sigma = 2n+1$. The case $\sigma = 2n$ corresponds to Case 3 while $\sigma = 2n+1$ implies that inequality (7.15) is not satisfied. When $t = l+1$, we obtain $\sigma \geq 2n + (l+1)(b-2)$. When $\sigma > 2n + (l+1)(b-2)$, (7.15) is again not satisfied.

We now consider the case where $\sigma = 2n + (l+1)(b-2)$, $t = l+1$ and $s = l$, implying that $\pi_1 = (1^{l-1}b^2(b+1)^{l-1})$. We will prove that π_1 is optimal and therewith prove the third part of *(iii)* of Proposition 7.3.2. Consider another (n,σ) partition π_2 with $i'_l = i'_{l+1} = b$ and let $s' = \min\{j \mid i'_j = b\} \leq l$ and $t' = \max\{j \mid i'_j = b\} > l$. The parts of π_1 and π_2 then satisfy

$$\begin{cases} i_j = 1 < b, \text{ if } 1 \leq j < l\,, \\ i_j = b \quad\;\;, \text{ if } l \leq j \leq l+1\,, \\ i_j = b+1\,, \text{ if } l+1 < j \leq n\,, \end{cases} \qquad \begin{cases} i'_j < b, \text{ if } 1 \leq j < s'\,, \\ i'_j = b, \text{ if } s' \leq j \leq t'\,, \\ i'_j > b, \text{ if } t' < j \leq n\,. \end{cases}$$

It follows that

$$Q_{\pi_2,\pi_1} = \frac{1}{n}\left((l-1) + \frac{1}{2}\left(\min(t'-s'+1,2)\right)\right) = \frac{1}{2}\,,$$

and therefore

$$\pi_1 = (1^{l-1}b^2(b+1)^{l-1}) \tag{7.22}$$

is an optimal strategy.

The aggregation of the results from Case 4 prove *(iii)* of Proposition 7.3.2. ∎

EXAMPLE – 7.3.7:

(i) Consider the $(6,20)_{\mathbf{L}}$ game, for which it holds that $b = 4$ and $c = 3$. In Figure 7.3 the Ferrers graph of each optimal strategy of the game is given. The optimal strategies satisfying $i_c = b$ are given by $(1^1 3^1 4^4)$, $(2^2 4^4)$, $(1^1 2^1 4^3 5^1)$, $(1^2 4^2 5^2)$ and $(1^2 4^3 6^1)$. Note that the fourth partition is of type (7.22), but as $c = l$ it is not a special case. As can be easily seen in the Ferrers graphs, these optimal strategies differ from each other by rearranging the $\Sigma_1 = 2$ dots that can be moved around freely. The remaining optimal strategies are those satisfying $i_{l+1} \geq b+1$, given by $(1^2 3^1 5^3)$, $(1^1 2^2 5^3)$, $(1^2 2^1 5^2 6^1)$, $(1^3 5^1 6^2)$ and $(1^3 5^2 7^1)$. These latter optimal strategies differ from each other by rearranging the $\Sigma_2 = 2$ free dots. Note that Σ_1 and Σ_2 are defined by (7.10).

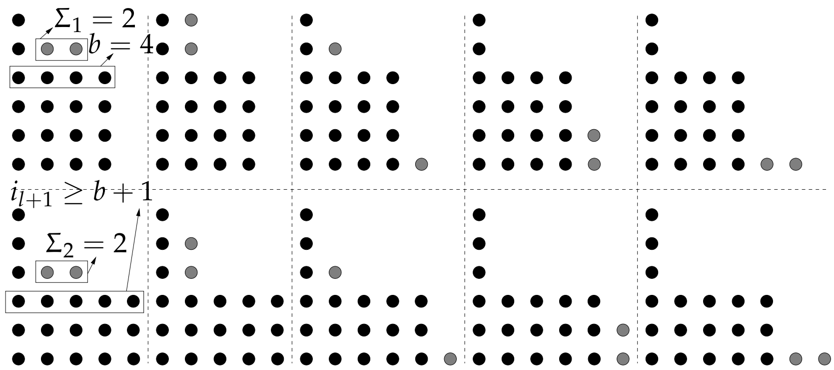


Figure 7.3: *Optimal strategies of the* $(6,20)_{\mathbf{L}}$ *game.*

(ii) Consider the $(12,32)_{\mathbf{L}}$ game. We obtain that $b = 3$, $c = 3$, $l(b+2) + b - 1 = 32 = \sigma$. The only $(12,32)$ partition satisfying $i_c = b$ is $\pi_1 = (1^2 3^{10})$ and when $\pi_2 = (1^5 3^1 4^6)$ it indeed holds that $Q_{\pi_2,\pi_1} > 1/2$. All optimal strategies are therefore those satisfying $i_{l+1} \geq b+1$, given by $(1^5 3^1 4^6)$, $(1^4 2^2 4^6)$, $(1^5 2^1 4^5 5^1)$, $(1^6 4^5 6^1)$ and $(1^6 4^4 5^2)$.

(iii) Consider the $(14,36)_{\mathbf{L}}$ game. We now obtain that $b = 3$, $c = 4$, $l(b+2) + b - 1 = 37 > \sigma$. There is one $(14,36)$ partition satisfying $i_c = b$, namely $(1^3 3^{11})$, and it is an optimal strategy. The other optimal strategies all satisfy $i_{l+1} \geq b+1$, and are given by $(1^6 2^1 4^7)$ and $(1^7 4^6 5^1)$.

(iv) Consider the $(4,22)_{\mathbf{L}}$ game. For the above 3 examples the optimal strategies satisfying $i_{l+1} \geq b$ always satisfied $i_{l+1} = b+1$. In general this is not true, as is indicated by the present example, for which it holds that $b = 7$ and for which the optimal strategy $(1^2 (10)^2)$ satisfies $i_{l+1} > b+1$. We do not explicitly specify the other optimal strategies for this game, as they are numerous.

◀

Case 5: $n = 2l+1 \wedge \sigma \neq 2n \wedge b \neq 1$.

All (n,σ) partitions π_1 for which $i_c = b$ or for which $i_{l+1} = b \wedge i_{l+2} = b+1$ are the only optimal strategies. The proof is completely analogous to the proof for n even. It follows directly that all optimal strategies π_1 must satisfy the condition $i_{l+1} = b$, and that such a strategy always exists. Secondly, it is evident that partitions of type

$$\pi_1 = (1^l b^1 (b+1)^l) \tag{7.23}$$

are optimal strategies and these only exist in $(2l+1, l(b+2)+b)_{\mathbf{L}}$ games. Finally, using Lemma 7.3.6, we obtain in a completely analogous way as in Case 4.3 that partitions of type (7.23) are the only possible optimal strategies that do not satisfy $i_c = b$, and that all (n,σ) partitions that satisfy $i_c = b$ are optimal. This proves (iv) of Proposition 7.3.2. Note that if $c = l+1$, partition (7.23) satisfies $i_c = b$ and is then not a special case. ∎

EXAMPLE – 7.3.8:
Consider the $(7,18)_{\mathbf{L}}$ game. We obtain that $b = c = 3$ and $\sigma = l(b+2)+b$. The optimal strategies are therefore given by $(1^3 3^1 4^3)$, $(1^2 3^4 4^1)$ and $(1^1 2^1 3^5)$, the first one being of type (7.23). ◀

As all possible $(n,\sigma)_{\mathbf{L}}$ games were covered in the above cases and for each case the games always had at least one optimal strategy, we have also proven Theorem 7.3.1. ∎

Using the obtained descriptions of the optimal strategies, we can now state the number of optimal strategies for any $(n,\sigma)_{\mathbf{L}}$ game, utilizing the function $p_n(M,N) = \sum_{i=0}^{N} p(N,M,N-i)p(N,n-M,i)$, which was already introduced in Proposition 7.3.4. This proposition is proven below, using the previously introduced values b and c, defined by (7.11), and Σ_1 and Σ_2 defined by (7.10). The number of optimal strategies in an $(n,\sigma)_{\mathbf{L}}$ game, here denoted as $\nu(n,\sigma)$, is then given by one of the cases below.

(i) $\sigma - n \leq \lfloor n/2 \rfloor \vee (n,\sigma) = (1,\sigma)$:
$\nu(n,\sigma) = 1$.

When $n = 1$ there is only one strategy, namely (σ). The result for $\sigma - n \leq \lfloor n/2 \rfloor$, which is equivalent to $b = 1$, follows from the result of Case 1.

(ii) $(n,\sigma) = (2,\sigma)$:
$\nu(n,\sigma) = \lfloor \frac{\sigma}{2} \rfloor$.

All strategies are optimal, this follows implicitly from the proofs of this subsection and this case is implicitly included in Proposition 7.3.4.

(iii) $(n,\sigma) = (n,2n)$:
$\nu(n,\sigma) = \lfloor \frac{n}{2} \rfloor + 1$.

This is immediately clear by counting the optimal strategies obtained in Case 3.

(iv) $(n,\sigma) = (2l,\sigma) \wedge \sigma = l(b+2)+b-1 \wedge \sigma \neq 2n \wedge b \neq 1$:
$\nu(n,\sigma) = p_n(l,\Sigma_2)$.

This corresponds to Case 4.2. We have to count the number of (n,σ) partitions for which $i_{l+1} \geq b+1$. We can construct all of them by starting with the Ferrers graph of $(1^l(b+1)^l)$ and distributing the remaining Σ_2 dots in all possible combinations to obtain all Ferrers graphs of (n,σ) partitions with $i_{l+1} \geq b+1$ (cfr. the bottom row of Figure 7.3).

(v) $(n,\sigma) = (2l,\sigma) \wedge \sigma = l(b+2) + b - 2 \wedge \sigma \neq 2n \wedge b \neq 1$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1) + p_n(l,\Sigma_2) + \lceil \frac{l-c}{l+1-c} \rceil$.

This corresponds to Cases 4.1 and 4.3, in the case that (7.22) is a possible strategy. Here, we have to count the number of (n,σ) partitions for which $i_c = b$, this is given by $p_n(c-1,\Sigma_1)$. We also have to count the number of (n,σ) partitions for which $i_{l+1} \geq b+1$, given by $p_n(l,\Sigma_2)$. Finally we also have to take into account the special case (7.22). Unless $c = l$, this partition has not yet been counted.

(vi) $(n,\sigma) = (2l,\sigma) \wedge \sigma < l(b+2) + b - 2 \wedge \sigma \neq 2n \wedge b \neq 1$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1) + p_n(l,\Sigma_2)$.

This corresponds to Cases 4.1 and 4.3, when (7.22) is not a possible strategy. This case and the previous case are combined into (iv) of Proposition 7.3.4.

(vii) $(n,\sigma) = (2l+1,\sigma) \wedge \sigma = l(b+2) + b \wedge \sigma \neq 2n \wedge b \neq 1$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1) + \lceil \frac{l+1-c}{l+2-c} \rceil$.

This corresponds to Case 5, in the case that (7.23) is a possible strategy. Here, we have to count the number of (n,σ) partitions for which $i_c = b$ and also the special partition $(1^l b^1 (b+1)^l)$, which has not yet been counted unless $c = l+1$.

(viii) $(n,\sigma) = (2l+1,\sigma) \wedge \sigma < l(b+2) + b \wedge \sigma \neq 2n \wedge b \neq 1$:
$\nu(n,\sigma) = p_n(c-1,\Sigma_1)$.

This corresponds to Case 5, when (7.23) is not a possible strategy. Note that the current case and the previous case are combined into (v) of Proposition 7.3.4. ∎

n \ $\sigma-n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48	52	56	61
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84	94	108	120	136	150	169
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141	164	192	221	255	291	333
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199	235	282	331	391	454	532
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248	300	364	436	522	618	733
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288	352	434	525	638	764	919
9	1	1	2	3	5	7	11	15	22	30	41	54	73	94	123	157	201	252	318	393	488	598	732	887	1076
10	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530	653	807	984	1204
11	1	1	2	3	5	7	11	15	22	30	42	56	76	99	131	169	219	278	355	445	560	695	863	1060	1303
12	1	1	2	3	5	7	11	15	22	30	42	56	77	100	133	172	224	285	366	460	582	725	905	1116	1380
13	1	1	2	3	5	7	11	15	22	30	42	56	77	101	134	174	227	290	373	471	597	747	935	1158	1436
14	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	175	229	293	378	478	608	762	957	1188	1478
15	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	230	295	381	483	615	773	972	1210	1508
16	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	296	383	486	620	780	983	1225	1530
17	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	384	488	623	785	990	1236	1545
18	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	489	625	788	995	1243	1556
19	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	626	790	998	1248	1563
20	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	791	1000	1251	1568
21	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1001	1253	1571
22	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1254	1573
23	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1574
24	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1575
25	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792	1002	1255	1575

Table 7.1: *Number of (n,σ) partitions.*

n \ $\sigma - n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	2	2	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	2	2	3	4	3	2	2	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	2	2	3	3	4	4	3	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	1	2	2	3	3	3	5	5	3	4	2	1	2	1	0	0	0	0	0	0	0	0	0	0
7	1	1	2	2	3	3	3	4	6	5	4	5	3	1	1	1	0	0	0	0	0	0	0	0	0
8	1	1	2	2	3	3	3	4	4	6	6	5	6	4	2	2	1	1	0	0	0	0	0	0	0
9	1	1	2	2	3	3	3	4	4	4	7	7	5	7	5	3	3	1	0	0	0	0	0	0	0
10	1	1	2	2	3	3	3	4	4	4	5	8	7	6	8	6	4	4	2	1	1	1	0	0	0
11	1	1	2	2	3	3	3	4	4	4	5	5	8	8	7	9	7	5	5	3	1	1	0	0	0
12	1	1	2	2	3	3	3	4	4	4	5	5	5	9	9	7	10	8	6	7	4	2	2	1	0
13	1	1	2	2	3	3	3	4	4	4	5	5	5	6	10	9	8	11	9	7	8	5	3	3	1
14	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	10	10	9	12	10	8	9	7	4	4
15	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	11	11	9	13	11	9	11	8	5
16	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	12	11	10	14	12	10	12	9
17	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	12	12	11	15	13	11	13
18	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	13	13	11	16	14	12
19	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	14	13	12	17	15
20	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	14	14	13	18
21	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	15	15	13
22	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	16	15
23	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	16
24	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	9
25	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	9

Table 7.2: *Number of optimal strategies for* $(n, \sigma)_{\mathbf{M}}$ *games.*

n \ $\sigma - n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	1	2	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	3	1	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
5	1	1	1	1	1	3	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
6	1	1	1	1	1	1	4	1	1	0	1	0	2	0	0	1	0	0	1	0	0	0	0	0	0
7	1	1	1	1	1	1	1	4	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
8	1	1	1	1	1	1	1	1	5	1	1	0	1	0	1	0	2	0	0	0	0	0	0	0	1
9	1	1	1	1	1	1	1	1	1	5	0	0	0	0	0	0	0	0	2	0	0	1	0	0	1
10	1	1	1	1	1	1	1	1	1	1	6	1	1	0	1	0	1	0	1	0	2	0	0	0	0
11	1	1	1	1	1	1	1	1	1	1	1	6	0	0	0	0	0	0	0	0	0	0	2	0	0
12	1	1	1	1	1	1	1	1	1	1	1	1	7	1	1	0	1	0	1	0	1	0	1	0	3
13	1	1	1	1	1	1	1	1	1	1	1	1	1	7	0	0	0	0	0	0	0	0	0	0	0
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	1	1	0	1	0	1	0	1	0	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	0	0	0	0	0	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	1	1	0	1	0	1	0	1
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	0	0	0	0	0	0	0
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	1	1	0	1	0	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	0	0	0	0	0
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	1	0	1
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	0	0	0
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	1	1
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	0
24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 7.3: *Number of optimal strategies for* $(n, \sigma)_{\mathbf{P}}$ *games.*

$\sigma - n$ / n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13
3	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
4	1	1	1	1	3	2	2	4	5	3	7	8	6	10	14	9	16	20	15	22	30	21	32	40	31
5	1	1	1	1	1	3	1	2	2	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	1
6	1	1	1	1	1	1	4	2	1	3	4	5	2	4	10	10	3	7	15	18	6	12	23	30	11
7	1	1	1	1	1	1	1	4	1	2	1	3	1	2	5	2	1	2	5	10	1	2	5	10	1
8	1	1	1	1	1	1	1	1	5	2	1	2	2	5	5	1	3	7	7	10	2	4	10	20	20
9	1	1	1	1	1	1	1	1	1	5	1	2	1	2	2	1	2	5	1	3	1	2	5	10	2
10	1	1	1	1	1	1	1	1	1	1	6	2	1	2	1	3	4	5	1	2	6	3	8	10	1
11	1	1	1	1	1	1	1	1	1	1	1	6	1	2	1	2	1	3	1	2	5	1	2	6	1
12	1	1	1	1	1	1	1	1	1	1	1	1	7	2	1	2	1	2	2	5	5	1	2	5	2
13	1	1	1	1	1	1	1	1	1	1	1	1	1	7	1	2	1	2	1	2	2	1	2	5	1
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	2	1	2	1	2	1	3	4	5	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8	1	2	1	2	1	2	1	3	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	2	1	2	1	2	1	2	2
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	9	1	2	1	2	1	2	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	2	1	2	1	2	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10	1	2	1	2	1
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	2	1	2	1
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	11	1	2	1
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	2	1
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	1
24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 7.4: *Number of optimal strategies for* $(n, \sigma)_{\mathbf{L}}$ *games.*

There is much pleasure to be gained from useless knowledge.
— BERTRAND RUSSEL

Their songs are on the whole very simple and mostly
follow the familiar theme of boy-being meets
girl-being beneath a silvery moon, which then
explodes for no adequately explored reason.
— DOUGLAS N. ADAMS

8 Categorizing Standard Collections

In this final chapter we discuss a way of partitioning standard collections of dice, introduced in Definition 3.2.1, with each dice having the same number of faces, into different categories based on their so-called street number. Originally, when dice-transitivity was not yet known to us, street numbers were introduced as a way of simplifying the discovery of the exact type of transitivity associated with a dice model. For a given number n and multiple types of known transitivity conditions, it was counted for how many standard collections of three dice with n faces the corresponding generated probabilistic relation satisfied the type of transitivity and these collections were, for each type of transitivity, partitioned into sets on the basis of their street number (see e.g. the three tables at the end of this chapter). When dice-transitivity was discovered, the street number remained useful as a tool to partition a set of standard collections consisting of dice with n faces. We will investigate this categorization in the present chapter and it will lead to some nice results and combinatorial problems. The main goal will be counting how many such standard collections have a given street number. Section 1 gives the general definition of a street number and Section 2 then investigates the street number of standard collections of 2 dice with n faces. In Section 3, the concept of a dual partition set is introduced, which will lead to an interesting method to construct "step triangles" by means of rectangles. Section 4 then briefly considers the street number of standard collections of 3 dice with n faces. It is indicated that it becomes very difficult to give general results concerning the number of standard collections of dice with n faces that have a given street number when considering collections consisting of 3 or more dice.

8.1 The street number

DEFINITION – 8.1.1: *Let $(A_1, A_2, \ldots, A_m)$ be a standard collection with each dice having n faces. Let $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ denote the sum of the numbers on dice $(A_1, A_2, \ldots, A_m)$. Let $M = n(mn+1)/2$ be the average sum per dice. The street number $s_{A_1,A_2,\ldots,A_m}$ of the collection is then defined by*

$$s_{A_1,A_2,\ldots,A_m} = \max(|\sigma_1 - M|, |\sigma_2 - M|, \ldots, |\sigma_m - M|) .$$

Standard collections of 2 (resp. 3) dice with n faces will be called standard n-duplets (resp. standard n-triplets). An immediate observation can be made about the boundaries of the street number.

PROPOSITION – 8.1.2: *The street number $s_{A_1,A_2,\ldots,A_m}$ of a standard collection of dice with n faces, is bounded by*

$$0 \leq s_{A_1,A_2,\ldots,A_m} \leq \frac{(m-1)n^2}{2} .$$

Proof:
The smallest sum of the integers on the faces of a dice that can be obtained is the sum of the first n integers, i.e. $n(n+1)/2$. This sum is $(m-1)n^2/2$ away from the average M. Similarly, the highest sum is the sum of the integers in

$\mathbb{N}[(m-1)n+1, mn]$, i.e. $n((2m-1)n+1)/2$, also at a distance of $(m-1)n^2/2$ from M. The lower bound can be obtained when n is even. We can then easily construct a standard collection of dice with n faces for which the values σ_i are equal to M. Indeed, attribute the mn integers to the faces of the dice as follows: dice A_i contains the integers taken from the sets $\{jm+i \mid j \in \mathbb{N}[0, \frac{n}{2}-1]\}$ and $\{nm-(jm+i-1) \mid j \in \mathbb{N}[0, \frac{n}{2}-1]\}$. For each dice A_i we then obtain as sum of the elements

$$\sigma_i = \sum_{j=0}^{\frac{n}{2}-1} (jm+i+nm-(jm+i-1)) = M\,.$$

■

We are able to state the street number of any standard collection of dice with n faces by means of the generated probabilistic relation $Q = [q_{ij}^{\mathbf{P}}]$. However, we start with a more general lemma.

LEMMA – 8.1.3: *Let $(A_1, A_2, \ldots, A_m)$ be a standard collection, with A_i having n_i faces and let $Q = [q_{ij}^{\mathbf{P}}]$ be the probabilistic relation generated by the collection when the dice are associated with independent random variables uniformly distributed over the elements of the respecive dice. The sum σ_i of the integers on the faces of dice A_i is then given by*

$$\sigma_i = \frac{n_i(n_i+1)}{2} + \sum_{j\neq i} n_i n_j q_{ij}^{\mathbf{P}}\,. \tag{8.1}$$

Proof:
We will count the number ν_i of couples (a,b) such that $a \in A_i, b \notin A_i$ and $n \geq a > b > 0$, where $n = \sum_j n_j$:

$$\nu_i = \#\{(a,b) \mid a \in A_i, b \notin A_i, n \geq a > b > 0\}\,.$$

As we are dealing with a standard collection, the number of couples (a_i, b_j), with $a_i \in A_i, b_j \in A_j$ and $a_i > b_j$ $(i \neq j)$, is given by $n_i n_j q_{ij}^{\mathbf{P}}$. Summing these values over all dice A_j, $j \neq i$, yields the desired value ν_i, namely $\nu_i = \sum_{j\neq i} n_i n_j q_{ij}^{\mathbf{P}}$.

We now obtain ν_i in another way. Suppose $A_i = (i_1, i_2, \ldots, i_{n_i})$, then for each integer i_j there are exactly $i_j - j$ integers on the faces of the other dice that are strictly smaller than i_j. Therefore, ν_i is also given by $(i_1-1)+(i_2-2)+\ldots+(i_{n_i}-n_i) = \sigma_i - n_i(n_i+1)/2$.

Combining both results yields the required equality for σ_i. ■

The above lemma can be immediately applied to prove the following proposition.

PROPOSITION – 8.1.4: *For a standard collection $(A_1, A_2, \ldots, A_m)$ of dice with n faces, the street number $s_{A_1,A_2,\ldots,A_m}$ is given by*

$$s_{A_1,A_2,\ldots,A_m} = n^2 \max_{1\leq i\leq m} \left| \sum_j (q_{ij}^{\mathbf{P}} - \frac{1}{2}) \right|\,. \tag{8.2}$$

Proof:
Using Lemma 8.1.3, we obtain $\sigma_i - M = n(n+1)/2 - n(mn+1)/2 + \sum_{j\neq i} n^2 q^{\mathbf{P}}_{ij}$ $= -(m-1)n^2/2 + \sum_{j\neq i} n^2 q^{\mathbf{P}}_{ij} = n^2 \sum_{j\neq i}(q^{\mathbf{P}}_{ij} - \frac{1}{2})$. As $q^{\mathbf{P}}_{ii} = 1/2$, there need not be any restriction on j. ∎

Using the payoff matrix $A = [a^1_{ij}]$ as defined in Chapter 5, we can restate the above proposition as follows.

PROPOSITION – 8.1.5: *The street number of a standard collection* $(A_1, A_2, \ldots, A_m)$ *of dice with* n *faces and corresponding payoff matrix* $A = [a^1_{ij}]$ *is given by*

$$s_{A_1,A_2,\ldots,A_m} = n^2 \max_i \left| \sum_j a^1_{ij} \right| . \tag{8.3}$$

Note that expression (8.3) makes use of a pseudonorm defined on the matrix A. It must also be noted that the concept of a street number is not applicable to the dice models used in Chapter 5, as the constraints on the dice differ from those in the current chapter.

For the case of standard triplets, Proposition 8.1.4 can be simplified as follows (using notations (2.1)).

COROLLARY – 8.1.6: *For a standard* n*-triplet* (A_1, A_2, A_3)*, it holds that the street number* s_{A_1,A_2,A_3} *equals* $n^2(\gamma_{123} - \alpha_{123})$.

Proof:
Applying Proposition 8.1.4, we immediately obtain that the street number is given by

$$\begin{aligned} s_{A_1,A_2,A_3} &= n^2 \max(\beta_{123} - \alpha_{123}, \gamma_{123} - \alpha_{123}, \gamma_{123} - \beta_{123}) \\ &= n^2(\gamma_{123} - \alpha_{123}) . \end{aligned}$$

∎

When only considering standard triplets of which the corresponding probabilistic relation is $T_{\mathbf{M}}$-transitive, we obtain the following interesting corollary.

COROLLARY – 8.1.7: *If the probabilistic relation generated by a standard* n*-triplet* (A_1, A_2, A_3) *is* $T_{\mathbf{M}}$*-transitive, then the parity of the corresponding street number* s_{A_1,A_2,A_3} *matches the parity of* n.

Proof:
This follows directly from Corollary 8.1.6 by noting that in the present case the equality $\alpha_{123} = 1 - \gamma_{123}$ holds (see (2.35)) and therefore also $s_{A_1,A_2,A_3} = n^2(2\gamma_{123} - 1)$, with $n^2\gamma_{123} \in \mathbb{N}$. ∎

We can partition all standard collections of m dice having n faces by means of their street number. The question now arises for each street number how many such collections have the given street number. This question turns out to be difficult to answer with closed formulas. We start by considering the case where $m = 2$.

8.2 Partitioning standard n-duplets

From Proposition 8.1.2 if follows that the street number s_{A_1,A_2} of a standard n-duplet is bounded from above by $n^2/2$. Note that the street number is an integer only when n is even. Suppose s_1, s_2 are the sum of the integers on dice A_1 resp. A_2 and let $M = n(2n+1)/2$ be the average sum as defined in Definition 8.1.1. It then holds that $s_2 - M = M - s_1$, as $s_1 + s_2 = n(2n+1)$, and the street number of any standard n-duplet is therefore given by $s_{A_1,A_2} = |M - s_1|$.

DEFINITION – 8.2.1: *The triangulated dice corresponding to a dice $A_1 = (i_1, i_2, \ldots, i_n)$, with $i_j > i_k$ when $j > k$, is given by*

$$A_1^{\tau} = (i_1 - 1, i_2 - 2, \ldots, i_n - n)\,.$$

Zeroes are therefore allowed on a triangulated dice. The name triangulated dice refers to the fact that the sum of the elements decreases by $n(n+1)/2$ and these are the well-known triangular numbers. Note that each dice with no identical elements has a unique corresponding triangulated dice and each triangulated dice corresponds to a unique dice (in which no two elements are equal).

THEOREM – 8.2.2: *The number of standard n-duplets of which the street number equals $s \leq n^2/2$ is given by*

$$p(n, n, M - s - n(n+1)/2)\,,$$

with p defined in Definition 1.4.6 and M the average sum of the integers.

Proof:
Note that $\sigma_1 = M - s$ denotes the sum of the integers on one of the dice of such a standard duplet. We need to count the number of ways to write σ_1 as the sum of the elements of a strictly increasing row of n integers taken from the set $\mathbb{N}[1, 2n]$. This is equivalent to counting the number of corresponding triangulated dice, in other words it is equivalent to the number of ways to write $\sigma_1 - n(n+1)/2$ as the sum of the elements of a nondecreasing row of n integers taken from $\mathbb{N}[0, n]$. This can be reformulated as the number of partitions of $\sigma_1 - n(n+1)/2$ into at most n parts, each less or equal to n, which can be written using Definition 1.4.6. ∎

8.3 Dual partition sets

By studying the Ferrers graph corresponding to an (n, σ) partition, the transformation done in the proof of Theorem 8.2.2 gave the inspiration to define a dual partition set, which consists of two partitions satisfying a certain condition, and we obtained a surprising connection between the two partitions composing a dual partition. Although it is unusual, in this section we allow partitions to have parts equal to 0. We start with the definition.

DEFINITION – 8.3.1: *The set of two partitions, in which we allow parts equal to 0, the first partition having m_1 parts, each part not exceeding m_2, and the second partition having m_2 parts, each part not exceeding m_1, such that the sum of the elements of both partitions equals $m_1 m_2$ and such that the Ferrers graph of both partitions can be combined, using a reflection and translation, to fill a rectangle of dimension $m_1 \times m_2$, is an (m_1, m_2) dual partition set.*

EXAMPLE – 8.3.2:
The partitions $(0, 0, 2, 3, 3, 4, 4)$ of 16 and $(2, 2, 3, 5)$ of 12 form a $(4, 7)$ dual partition set. Indeed, with $m_1 = 4$ and $m_2 = 7$, the corresponding Ferrers graphs can be combined to fill a rectangle with dimensions 4×7, as can be seen in Figure 8.1. We see that the Ferrers graph corresponding to the partition of 16 is first reflected around the second diagonal, after which it perfectly fits to construct a 4×7 rectangle with the second Ferrers graph.

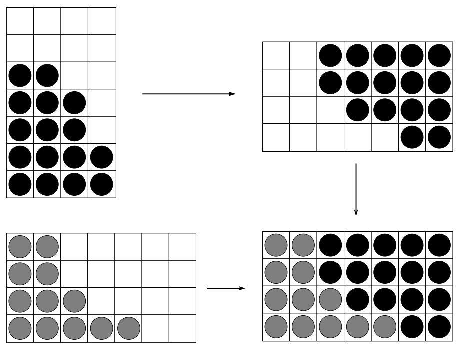


Figure 8.1: *An example of a dual partition set.*

◀

THEOREM – 8.3.3: *For two partitions $(i_1, i_2, \ldots, i_{m_1})$ and $(j_1, j_2, \ldots, j_{m_2})$, in which we allow parts with value zero, that together compose an (m_1, m_2) dual partition set, it holds that the number sets $\{i_k + k | k \in \mathbb{N}[1, m_1]\}$ and $\{j_k + k | k \in \mathbb{N}[1, m_2]\}$ form a set partition of the number set $\mathbb{N}[1, m_1 + m_2]$.*

Proof:
As both partitions form a dual partition set, we have that $i_k \leq m_2$ and $j_l \leq m_1$, which implies that $i_k + k \leq m_1 + m_2$ and $j_l + l \leq m_1 + m_2$, for $k \in \mathbb{N}[1, m_1]$ and $l \in \mathbb{N}[1, m_2]$. We now prove that $i_k + k \neq j_l + l, \forall (k \in \mathbb{N}[1, m_1], l \in \mathbb{N}[1, m_2])$. Crucial to the proof is the fact that the rows $(i_k)_{k \in \mathbb{N}[1, m_1]}$ and $(j_l)_{l \in \mathbb{N}[1, m_2]}$ are nondecreasing.

For any k such that $j_k \neq 0$ and $j_k \neq m_1$, it holds that $i_{j_k} + j_k \leq j_k + k - 1$ and $i_{j_k + 1} + j_k + 1 \geq j_k + k + 1$. Indeed, when considering the $m_1 \times m_2$ rectangle as a matrix with as elements black dots and empty space, we see that if $i_{j_k} \geq k$, it would mean that there is a black dot on position (i_{j_k}, k) of the matrix, which would mean that the number of empty squares in column k is less than j_k. We

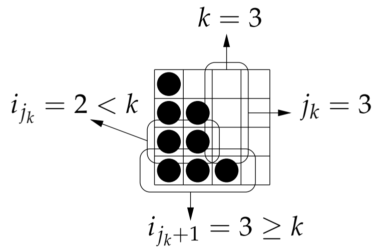


Figure 8.2: *Illustration of the proof.*

must therefore have that $i_{j_k} < k$. On the other hand, if there were no dot on position (i_{j_k+1}, k), we would have that the number of empty squares in column k is higher than j_k. This is illustrated in Figure 8.2, which shows a $(4, 4)$ dual partition set. The first partition is given by $(i_1, i_2, i_3, i_4) = (1, 2, 2, 3)$ (counting the black dots row by row) and the second by $(j_1, j_2, j_3, j_4) = (0, 1, 3, 4)$ (counting the empty squares column by column). One indeed verifies that $i_{j_k} < k$ and $i_{j_k+1} \geq k$, for $k \in \{2, 3\}$. As the row $(i_l + l)_{l\in\mathbb{N}_0}$ is (strictly) increasing, the above implies that $i_l + l \neq j_k + k$, with $l \in \mathbb{N}[1, m_1]$. As this holds for any k satisfying the mentioned constraints, we have that $i_l + l \neq j_k + k$ for $l \in \mathbb{N}[1, m_1]$ and k such that $j_k \neq 0 \wedge j_k \neq m_1$. We still need to consider the 2 special cases. When $j_k = 0$ this implies that $i_1 \geq k$ or equivalently $i_1 + 1 \geq j_k + k + 1$. When $j_k = m_1$ this implies that $i_{m_1} < k$ or equivalently $i_{m_1} + m_1 < j_k + k$. In both special cases for j_k we again obtain that $i_l + l \neq j_k + k, \forall\, l \in \mathbb{N}[1, m_1]$. As the total number of integers is given by $m_1 + m_2$ and all integers are different and contained in the set $\mathbb{N}[1, m_1 + m_2]$, we obtain the desired result. ∎

COROLLARY – 8.3.4: *For any two dice of a standard duplet it holds that their corresponding triangulated dice form a dual partition set.*

The above theorem shows a nice property of "step triangles" (triangles constructed using squares of equal size). Suppose we have a set of n rectangles of height 1 and each of different length, with the length an integer smaller or equal to n. Take any $m \leq n$ rectangles out of this set and order them in increasing order of length. Put these rectangles one beneath the other, so that the left-hand side of each rectangle is located one position more to the left than its predecessor (if there is one) and such that a rectangle located below another has a higher length. The remaining $n - m$ rectangles can now be used, in a unique way, to construct a "step triangle" with surface $n(n + 1)/2$. Again order the remaining rectangles in increasing order of length and stack them on top of the m rectangles, from left to right (starting at the appropriate position). As is so often the case, this is best explained using an illustration, see Figure 8.3. In the figure, we see two examples, each with 8 rectangles. In the first example (resp. second example), 3 (resp. 4) rectangles, of length 2, 5 and 7 (resp. 1, 4, 5 and 7) were chosen. Note that the fact that this construction always works follows directly from Theorem 8.3.3. Also note that one can try this at home after utilizing scissors to make the rectangles out of paper. Be sure not to cut yourself (on the paper or the scissors).

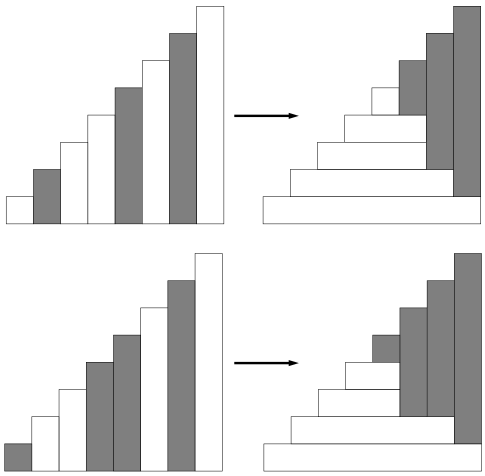


Figure 8.3: *Constructing triangles with rectangles.*

8.4 Partitioning standard n-triplets

The proof of the result in this section makes use of the well-known values $Q(n,k)$, which are expressable using the function p defined in Definition 1.4.6.

PROPOSITION – 8.4.1: *The number of partitions of n into exactly k distinct parts, denoted as $Q(n,k)$, is given by*

$$Q(n,k) = p(n - k(k+1)/2, n, n - k(k+1)/2)\,. \tag{8.4}$$

Proof:
For $n < k(k+1)/2$ the proof is trivial. Suppose now that $n \geq k(k+1)/2$ and let $\pi = (i_1, i_2, \ldots, i_k)$ be a partition of $n - k(k+1)/2$ into at most k parts ($i_j = 0$ is therefore allowed). The partition $\pi' = (1 + i_1, 2 + i_2, \ldots, k + i_k)$ is then a partition of n into exactly k (non-negative) distinct parts. It is now obvious that there is a one-to-one correspondence between partitions of type π and those of type π', which proves (8.4). ■

When going from two dice to three dice, counting the number of standard collections with equal number of faces that have a given street number gets a lot more complicated. Already for the case of two dice, the number of standard n-duplets having a given street number is determined using a generating function for partitions, so it is not surprising that in the case of three dice it becomes difficult to make general statements about the number of standard

n-triplets having a given street number. We can, however, state some exact results for street numbers that are high enough. Note that for $m = 3$, M is given by $n(3n+1)/2$.

PROPOSITION – 8.4.2: *The number $\nu(s)$ of standard n-triplets with a given street number $s \in \mathbb{N}[n^2 - \lfloor n/2 \rfloor, n^2]$ is given by*

$$\nu(s) = Q(M-s,n)\left(\binom{2n}{n} - Q(M-s,n) - 2\sum_{i=0}^{n^2-s-1} p(i)\right), \tag{8.5}$$

where $p(i)$ and $Q(i,j)$, are defined in Definition 1.4.1, resp. 8.4.1.

Proof:
The maximal street number, n^2 is obtained when one of the dice contains the set of numbers $\mathbb{N}[1,n]$ or $\mathbb{N}[2n+1,3n]$. The number of standard n-triplets with one dice holding the numbers $\mathbb{N}[1,n]$ is given by $\binom{2n}{n}/2$. The same holds for those having one dice with the numbers $\mathbb{N}[2n+1,3n]$. There is one standard n-triplet that belongs to both types of the above triplets, namely the one consisting of the dice $\mathbb{N}[1,n]$, $\mathbb{N}[n+1,2n]$ and $\mathbb{N}[2n+1,3n]$. The total number of standard n-triplets with street number n^2 is thus given by $\binom{2n}{n} - 1$. As for $s = n^2$ we have that $M - s = n(n+1)/2$, (8.5) is already proven for $s = n^2$.

We will now obtain the other results. First note that if the sum of the elements of one dice of a standard n-triplet equals $M - s$, with $s \in \mathbb{N}[n^2 - \lfloor n/2 \rfloor]$, then the sum of the elements of any other dice of the standard n-triplet is strictly greater than $M - s$. Indeed, suppose the sums of the 3 dice are resp. s_1, s_2, s_3 and assume $s_2 \leq s_1 = M - s$. As $s_1 + s_2 + s_3 = 3M$, it then holds that $s_3 \geq M + 2s \geq M + 2n^2 - n = n(7n-1)/2 > n(5n+1)/2$, which is impossible as the maximum sum on a dice is $n(5n+1)/2$.

As was introduced in Proposition 8.4.1, the number of partitions of $M - s$ into exactly n distinct parts is given by $Q(M-s,n)$. The number of n-triplets for which the sum of the elements of a dice equals $M - s$ is then given by $Q(M-s,n)C_{2n}^2/2$. For these triplets, the street number is already at least s. The number of these triplets for which the street number is strictly greater than s is given by $Q(M-s,n)\sum_{i=0}^{n^2-s-1} p(i)$. Indeed, this is the sum of the number of partitions of $M + n^2 - i$ into exactly n distinct parts, for all $i \in \mathbb{N}[0, n^2 - s - 1]$, multiplied by the number of partitions of $M - s$ into exactly n distinct parts. As follows from the previous paragraph, we have herewith counted all such standard n-triplets.

A similar reasoning can be used to obtain the number of n-triplets with street number $s \in \mathbb{N}[n^2 - \lfloor n/2 \rfloor, n^2]$ for which the highest sum of the elements of a dice equals $M + s$. This is also given by $Q(M-s,n)(Q(M-s,n)C_{2n}^n/2 - \sum_{i=0}^{n^2-s-1} p(i))$.

When we sum both types of n-triplets, we have counted the n-triplets which have a highest sum of $M + s$ and a lowest sum of $M - s$ twice. There are exactly $Q(M-s,n)^2$ such triplets.
Combining the above results we obtain (8.5). ∎

The condition $s \geq n^2 - \lfloor n/2 \rfloor$ is essential in the above proposition, because when we do not impose this condition, the integers on a dice of a standard n-triplet for which the sum equals $M + s$ aren't necessarily all different from the integers on a dice of a standard n-triplet for which the sum equals $M - s$. It is however very difficult to obtain a closed form for the number of triplets that are counted more than once when the condition $2(n^2 - s) \leq n$ is not imposed.

We end this chapter with three tables that list (for $n = 5$, $n = 6$ and $n = 7$), for each street number, the number of standard n-triplets with that specific street number (these are given in the column named U_D). Moreover, the number of these triplets that satisfy $T_{\mathbf{M}}$-transitivity, resp. $T_{\mathbf{P}}$-transitivity, resp. cycle-transitivity w.r.t. the upper bound function $U_{PD}(\alpha, \beta, \gamma) = \alpha + \gamma - \alpha\gamma$, and also the number of cyclic triplets among these (i.e. those not satisfying weak stochastic transitivity) are listed. These numbers can be found in the columns named $U_{\mathbf{M}}$, resp. $U_{\mathbf{P}}$, resp. U_{PD}, resp. $\neg U_{ws}$. Note that the result from Corollary 8.1.7 is represented by the occurrences of 0 in the columns named $U_{\mathbf{M}}$.

street nr.	U_D	$U_{\mathbf{M}}$	$U_{\mathbf{P}}$	U_{PD}	$\neg U_{ws}$
0	305	0	305	305	305
1	1891	1164	1891	1891	727
2	3640	0	3640	3640	434
3	5346	2919	5346	5346	178
4	6723	0	6721	6723	56
5	8018	3843	8010	8018	16
6	8773	0	8759	8773	8
7	9468	3775	9429	9466	8
8	9582	0	9527	9582	0
9	9671	3444	9556	9671	0
10	9237	0	9037	9236	0
11	8836	2551	8498	8828	0
12	8034	0	7674	8031	0
13	7329	2044	6838	7328	0
14	6247	0	5751	6247	0
15	5520	1579	4967	5520	0
16	4448	0	3875	4448	0
17	3639	1028	3124	3639	0
18	2743	0	2275	2743	0
19	2193	520	1801	2193	0
20	1591	0	1255	1591	0
21	1176	330	987	1176	0
22	724	0	587	724	0
23	492	140	415	492	0
24	249	0	193	249	0
25	251	251	251	251	0
total	126126	23588	120712	126111	1732

Table 8.1: *Standard* 5*-triplets partitioned according to the street number.*

street nr.	U_D	$U_{\mathbf{M}}$	$U_{\mathbf{P}}$	U_{PD}	$\neg U_{ws}$
0	4331	2421	4331	4331	1910
1	25287	0	25287	25287	4342
2	50237	24693	50237	50237	2616
3	73384	0	73383	73384	1091
4	95614	44344	95606	95611	414
5	114269	0	114255	114267	222
6	132042	51582	132032	132042	84
7	144640	0	144617	144640	24
8	156303	54425	156287	156303	2
9	162749	0	162702	162746	0
10	167822	47880	167709	167820	0
11	167433	0	167108	167433	0
12	167180	46128	166812	167174	0
13	160600	0	159696	160598	0
14	155077	35272	153689	155070	0
15	145046	0	143225	145043	0
16	135878	30820	133347	135878	0
17	122813	0	119610	122809	0
18	112547	22237	108318	112542	0
19	98715	0	94370	98697	0
20	87840	17584	82728	87832	0
21	74955	0	70190	74952	0
22	64411	11193	59249	64410	0
23	52876	0	47937	52876	0
24	45103	10112	40355	45103	0
25	35141	0	30827	35141	0
26	28619	5270	24752	28619	0
27	21751	0	18537	21751	0
28	17075	3562	14307	17075	0
29	12176	0	10219	12176	0
30	9719	1828	7929	9719	0
31	6275	0	5399	6275	0
32	4529	1176	3796	4529	0
33	2739	0	2236	2739	0
34	1836	504	1542	1836	0
35	921	0	711	921	0
36	923	923	923	923	0
total	2858856	411954	2794258	2858789	10705

Table 8.2: *Standard 6-triplets partitioned according to the street number.*

street nr.	U_D	$U_{\mathbf{M}}$	$U_{\mathbf{P}}$	U_{PD}	$\neg U_{ws}$
0	63261	0	63261	63261	63261
1	379871	172192	379871	379871	207679
2	752219	0	752219	752219	190296
3	1115120	473938	1115120	1115120	117316
4	1456742	0	1456742	1456742	60992
5	1780217	688625	1780213	1780217	30437
6	2070447	0	2070441	2070447	14593
7	2335868	796350	2335847	2335868	6847
8	2558392	0	2558364	2558392	3170
9	2752446	820378	2752366	2752442	1585
10	2898796	0	2898666	2898791	818
11	3015103	792093	3014782	3015086	413
12	3082397	0	3081958	3082379	152
13	3122328	734360	3121458	3122316	76
14	3113813	0	3112692	3113807	42
15	3083208	645895	3081186	3083199	28
16	3009580	0	3007050	3009580	14
17	2920195	565538	2915237	2920190	14
18	2794201	0	2787342	2794177	0
19	2660664	473966	2650595	2660649	0
20	2497158	0	2483939	2497144	0
21	2335675	393498	2316189	2335653	0
22	2152098	0	2126468	2152088	0
23	1976970	313856	1943122	1976956	0
24	1788753	0	1755023	1788742	0
25	1615145	254012	1574322	1615138	0
26	1433190	0	1385980	1433181	0
27	1271341	185401	1219902	1271325	0
28	1107985	0	1053837	1107970	0
29	964790	142666	912484	964756	0
30	823658	0	769150	823640	0
31	703905	100080	651048	703897	0
32	586630	0	537929	586627	0
33	491164	72116	446735	491163	0
34	400686	0	358794	400686	0
35	328692	51747	292647	328692	0
36	260290	0	227930	260290	0
37	208981	34830	182373	208981	0
38	159417	0	137344	159417	0
39	124931	18648	107200	124931	0
40	92923	0	78573	92923	0
41	70258	12782	59714	70258	0
42	50528	0	41845	50528	0
43	37265	6608	31763	37265	0
44	23818	0	20351	23818	0
45	17066	4284	14193	17066	0
46	10263	0	8269	10263	0
47	6852	1848	5730	6852	0
48	3429	0	2637	3429	0
49	3431	3431	3431	3431	0
total	66512160	7759142	65684332	66511863	697733

Table 8.3: *Standard 7-triplets partitioned according to the street number.*

Not everything that counts can be counted,
and not everything that can be counted counts.
— ALBERT EINSTEIN

Conclusion

This work consists of four related parts, divided into eight chapters. A first part introduces the framework of cycle-transitivity, developed by De Baets et al. It is shown that this framework is ideally suited for describing and comparing forms of transitivity of probabilistic relations. Not only does it encompass most already known concepts of transitivity, it is also ideally suited to describe new types of transitivity that are encountered in this work (such as isostochastic transitivity and dice-transitivity). The author made many non-trivial and sometimes vital contributions to the development of this framework.

A second part consists of the development and study of a new method to compare random variables. This method, which bears the name generalized dice model, was developed by De Meyer et al. and De Schuymer et al., and can be seen as a graded alternative to the well-known concept of first degree stochastic dominance.

A third part involves the determination of the optimal strategies of three game variants that are closely related to the developed comparison scheme. The definitions of these variants differ from each other solely by the copula that is used to define the payoff matrix. It turns out however that the characterization of the optimal strategies, done by De Schuymer et al., is completely different for each variant.

A last part includes the study of some combinatorial problems that originated from the investigation of the transitivity of probabilistic relations obtained by utilizing the developed method to compare random variables. The study, done by De Schuymer et al., includes the introduction of some new and interesting concepts in partition theory and combinatorics.

A more thorough discussion, in which each section of this work is taken into account, can be found in the overview at the beginning of this manuscript.

Although this work is oriented towards a mathematical audience, the introduced concepts are immediately applicable in practical situations. Firstly, the framework of cycle-transitivity provides an easy means to represent and compare obtained probabilistic relations. Secondly, the generalized dice model delivers a useful alternative to the concept of stochastic dominance for comparing random variables. Thirdly, the considered dice games can be viewed in an economical context in which competitors have the same resources and alternatives, and must choose how to distribute these resources over their alternatives.

Finally, it must be noted that this work still leaves opportunities for future research. As immediate candidates we see, firstly the investigation of the transitivity of generalized dice models in which the random variables are pairwisely coupled by a different copula. Secondly, the characterization of the transitivity of higher-dimensional dice models, starting with dimension 4. Thirdly, the study of the applicability of the introduced comparison schemes in areas such as market efficiency, portfolio selection, risk estimation, capital budgeting, discounted cash flow analysis, etc.

Samenvatting

Het Vergelijken van Toevalsveranderlijken vanuit een Speltheoretisch Perspectief

Het vergelijken van toevalsveranderlijken met behulp van een probabilistische relatie is het centrale thema in dit werk. De vergelijkingsmethode die doorheen dit werk wordt gebruikt, wordt in Hoofdstuk 3 ingevoerd. Essentieel voor de methode is de probabiliteit dat een zekere toevalsveranderlijke een grotere waarde aanneemt dan een andere toevalsveranderlijke. In speltheoretische context (wanneer de toevalsveranderlijken met spelers worden geassocieerd), kan deze probabiliteit ook gedefinieerd worden als de kans dat een speler wint van een andere speler. De toevalsveranderlijken worden dus vergeleken vanuit een speltheoretisch perspectief. Meer nog, het is heel natuurlijk om de elementen van de probabilistische relatie, die verkregen worden door deze vergelijkingsmethode te gebruiken, te transformeren naar de elementen van een opbrengstmatrix waarmee de spellen die in Hoofdstukken 5 en 7 worden behandeld, geassocieerd worden.

In Hoofdstuk 1 worden enkele basisconcepten ingevoerd die gebruikt zullen worden in de daarop volgende hoofdstukken. In de eerste sectie worden de relationele concepten en de eraan verwante onderwerpen die in dit werk aan bod zullen komen, ingevoerd. De tweede sectie introduceert distributiefuncties en geeft hun verband met copula's, daarnaast wordt ook nog een voor dit werk belangrijke opmerking in verband met toevalsvectoren vermeld. In de derde sectie worden dan de speltheoretische concepten ingevoerd die in Hoofdstukken 5 en 7 van belang zullen zijn. De laatste sectie introduceert dan enkele basisconcepten uit de partitietheorie die van toepassing zijn in Hoofdstukken 5, 7 en 8.

Hoofdstuk 2 introduceert het raamwerk van de cykeltransitiviteit, dat ideaal geschikt is voor het beschrijven en vergelijken van diverse vormen van transitiviteit. Aangezien vele probabilistische relaties aan bod zullen komen in dit werk, inclusief relaties die niet noodzakelijk transitief zijn in de strikte zin (zoals dobbeltransitieve relaties), is er een middel nodig om deze relaties op een uniforme wijze voor te stellen. Het kader van de cykeltransitiviteit is hiervoor ideaal aangezien de transitiviteit van diverse types van probabilistische relaties in dit model voorgesteld kunnen worden. Cykeltransitiviteit is een manier om een 3-dimensionale probabilistische relatie te beschrijven aan de hand van de cyclische evaluatie van de gewichten van de corresponderende gewogen graaf en ze is alomtegenwoordig in dit werk. In de eerste twee secties wordt dit raamwerk ontwikkeld. Secties drie en vier beschouwen dan de cykeltransitieve representatie van de twee meest bekende types van transitiviteit, met name de T-transitiviteit en de stochastische transitiviteit. Tijdens deze discussie zullen nieuwe types transitiviteit, zoals partiële g-stochastische transitiviteit en isostochastische transitiviteit aan bod komen. De laatste sectie bespreekt in het kort een alternatieve representatie van probabilistische relaties door gebruik te maken van de zogenaamde symmetrische opbrengstrelaties.

In Hoofdstuk 3 wordt het discrete dobbelsteenmodel ingevoerd en wordt de daarbijhorende karakteristieke vorm van cykeltransitiviteit, dobbeltransitiviteit genaamd, bepaald. Dit model kan gebruikt worden voor het paarsgewijs vergelijken van lijsten van natuurlijke getallen en zo een lijst kan ook aanzien worden als een dobbelsteen waarbij op elk vlak een element uit de lijst geschreven is. Dit model zal met de dobbelsteenrepresentatie gebruikt worden in Hoofdstuk 5 als het model waarin een interessante klasse van spellen, zogenaamde (n, σ) dobbelspellen, wordt gedefinieerd. Het is ook de basis van de algemene vergelijkingsmethode voor toevalsveranderlijken die zal worden ontwikkeld in Hoofdstukken 4 en 6. De eerste twee secties introduceren het dobbelsteenmodel terwijl de derde sectie het belang van de zogenaamde standaardcollectie van lijsten aanduidt. In de vierde sectie wordt dan bewezen dat dobbeltransitiviteit de karakteristieke transitiviteit is die door een 3-dimensionaal dobbelsteenmodel wordt gegenereerd. In de vijfde sectie wordt een zeer natuurlijke vraag omtrent de aard van dobbeltransitiviteit beantwoord en de relatie met de probabilistische som wordt duidelijk gemaakt. Daarenboven wordt bewezen dat eender welke 3-dimensionale dobbeltransitieve relatie met rationale elementen gegenereerd kan worden door een 3-dimensionaal dobbelsteenmodel bestaande uit hoogstens zeven zogenaamde blokken, wat impliceert dat het dobbelsteenmodel dat gekozen wordt om een gegeven 3-dimensionale dobbeltransitieve relatie te genereren een veel eenvoudigere structuur kan hebben dan een willekeurig dobbelsteenmodel. De transitiviteit van dobbelsteenmodellen met een hogere dimensie wordt onderzocht in de zesde sectie. Er wordt bewezen dat niet alle 4-dimensionale dobbeltransitieve relaties met rationale elementen gegenereerd kunnen worden door een 4-dimensionaal dobbelsteenmodel, wat impliceert dat dobbeltransitiviteit niet langer de karakteristieke transitiviteit is van hogerdimensionale dobbelsteenmodellen. Ondanks verschillende ondernomen pogingen om de karakteristieke transitiviteit te vinden van 4-dimensionale dobbelsteenmodellen, zijn we hierin niet geslaagd. Er zal echter wel worden aangetoond dat alle 4-dimensionale $T_{\mathbf{M}}$-transitieve probabilistische relaties met rationale elementen gegenereerd kunnen worden door een 4-dimensionaal dobbelsteenmodel.

In Hoofdstuk 4 wordt de in het voorgaande hoofdstuk geïntroduceerde methode gegeneraliseerd om zo een mathematisch werkmiddel te bekomen voor het paarsgewijs vergelijken van toevalsveranderlijken. De eerste sectie generaliseert het dobbelsteenmodel. Afhankelijk van het feit dat discrete of continue toevalsveranderlijken worden vergeleken, worden deze modellen gegeneraliseerde discrete dobbelsteenmodellen of gegeneraliseerde continue dobbelsteenmodellen genoemd. Deze modellen genereren allen probabilistische relaties en wanneer onafhankelijke toevalsveranderlijken worden vergeleken voldoen deze relaties allen ten minste aan de dobbeltransitiviteit. De noodzaak om paarsgewijs toevalsveranderlijken te vergelijken komt frequent voor in het vakgebied van "decision making". Deze nieuwe methode om onafhankelijke toevalsveranderlijken paarsgewijs te vergelijken biedt een geschaald alternatief voor het concept van stochastische dominantie, wat ook zeer populair is in het zonet genoemde vakgebied, bijvoorbeeld in economische applicaties. In Hoofdstuk 4 wordt de nadruk gelegd op het vergelijken van toevalsveranderlijken door ze als onafhankelijk te beschouwen. De vergelijkingsmethode gebruik makend van een copula die verschilt van de $T_{\mathbf{P}}$-copula wordt pas bestudeerd in Hoofdstuk 6. In de tweede sectie van Hoofdstuk 4 wordt bewezen dat de karakteristieke transitiviteit van de gegeneraliseerde dobbelsteenmodellen met onafhankelijke toevalsveranderlijken de dobbeltransitiviteit blijft. Het overblijvende deel van dit hoofdstuk behandelt dan meer specifieke families van toevalsveranderlijken. De nadruk wordt gelegd op de types van cykeltransitiviteit van de relaties die door deze specifieke modellen gegenereerd kunnen worden. Nieuwe types van transitiviteit en interessante connecties met het gebied van de copula's worden onthuld. De derde sectie handelt over de algemene klasse van dobbelsteenmodellen die bestaan uit onafhankelijke toevalsveranderlijken waarvan de cumulatieve distributiefuncties willekeurige translaties van dezelfde cumulatieve distributiefunctie zijn. De vierde sectie bestudeert dan verschillende éénparametrische families van toevalsveranderlijken en verschillende types transitiviteit, inclusief multiplicatieve transitiviteit en specifieke vormen van isostochastische transitiviteit, worden hierbij ontmoet. De vijfde sectie behandelt dobbelsteenmodellen bestaande uit normaalgedistribueerde onafhankelijke toevalsveranderlijken waarbij zowel de verwachtingswaarde als de variantie als vrije parameters worden beschouwd. Er wordt aangetoond dat gematigde stochastische transitiviteit de karakteristieke transitiviteit is van zulke 3-dimensionale modellen. In de laatste sectie worden dobbelsteenmodellen bestaande uit uniform verdeelde onafhankelijke toevalsveranderlijken met overlappende dragers behandeld en de bijhorende karakteristieke transitiviteit voor zulke 3-dimensionale modellen wordt bepaald. Dit blijkt een type van g-stochastische transitiviteit te zijn.

Hoofdstuk 5 is gewijd aan het achterhalen van de optimale strategieën van een klasse van spellen die nauw verband houdt met het dobbelsteenmodel dat in Hoofdstuk 3 werd geïntroduceerd. De beschouwde spellen zijn symmetrische matrixspellen gespeeld door twee spelers die beschikken over een collectie van dobbelstenen met een vast aantal vlakken en met op elk vlak een strikt positief geheel getal zodanig dat deze getallen sommeren tot een vastgelegde som. De spellen die in Hoofdstuk 7 zullen worden behandeld houden nauw verband met deze die in Hoofdstuk 5 worden behandeld. In hun statistische interpretatie verschillen ze enkel door de gebruikte copula voor het bepalen van de opbrengstmatrix. Hoofdstuk 5 beschouwt de spellen waarbij de $T_{\mathbf{P}}$-copula wordt gebruikt. Het vinden van de optimale strategieën voor deze spelvariant houdt daarom nauw verband met het vergelijken van onafhankelijke uniform verdeelde toevalsveranderlijken. In de eerste drie secties van Hoofdstuk 5 wordt een volledige beschrijving van de beschouwde spelvariant gegeven. In de vierde sectie worden dan de antwoorden op de volgende vragen met betrekking tot het karakter van de optimale strategieën geformuleerd: welke (n, σ) dobbelspellen bezitten optimale strategieën, hoe zien deze optimale strategieën eruit en met hoeveel zijn ze ? In de vijfde sectie worden de resultaten uit de voorgaande sectie bewezen.

In Hoofdstuk 6 wordt de methode voor het paarsgewijs vergelijken van toevalsveranderlijken, die in Hoofdstuk 4 werd geïntroduceerd, toegepast door de toevalsveranderlijken paarsgewijs te koppelen met een van de $T_{\mathbf{P}}$-copula verschillende copula. De nadruk wordt gelegd op de twee extreme copula's. In de eerste sectie wordt een alternatieve methode, die gebruik maakt van de zogenaamde diagonaalformule, voor het bekomen van de probabilistische relatie gegenereerd door een dobbelsteenmodel geïntroduceerd. Er wordt ook een equivalente representatie gebruik makend van geordende lijsten bekomen voor dobbelsteenmodellen bestaande uit discrete toevalsveranderlijken die paarsgewijs gekoppeld worden door één van de extreme copula's. Deze representatiewijze vormt de link met de spelvarianten die in Hoofdstuk 7 worden bestudeerd en deze geordende lijstrepresentatie wordt ook gebruikt in de tweede sectie om de karakteristieke transitiviteit van deze modellen te bepalen. Er wordt bewezen dat $T_{\mathbf{L}}$-transitiviteit (resp. partiële min-stochastische transitiviteit) de karakteristieke transitiviteit is van 3-dimensionale dobbelsteenmodellen bestaande uit discrete toevalsveranderlijken die voor het vergelijken paarsgewijs gekoppeld worden met de $T_{\mathbf{M}}$-copula (resp. $T_{\mathbf{L}}$-copula). Er wordt ook bewezen dat voor geen van beide beschouwde dobbelsteenmodellen, de specifieke transitiviteit behouden blijft als karakteristieke transitiviteit wanneer hogerdimensionale dobbelsteenmodellen worden beschouwd. De derde sectie behandelt dan continue dobbelsteenmodellen waarin één van de twee extreme copula's wordt gebruikt voor het vergelijken van de toevalsveranderlijken. Voor beide types dobbelsteenmodellen wordt een interessante manier voor het bepalen van de probabilistische relatie via de grafieken van de marginale cumulatieve distributiefuncties (corresponderende met de beschouwde toevalsveranderlijken) bekomen.

Hoofdstuk 7 behandelt twee spelvarianten die buiten de gebruikte copula op identieke wijze gedefinieerd zijn als de spelvariant uit Hoofdstuk 5. De eerste sectie geeft een kort overzicht van de drie spelvarianten die in dit werk aan bod komen. De twee daaropvolgende secties behandelen dan de spelvariant waarin de $T_{\mathbf{M}}$-copula wordt gebruikt. In de tweede sectie worden de resultaten omtrent de optimale strategieën gebundeld en deze worden dan bewezen in de derde sectie. Secties vier en vijf behandelen de spelvariant waarin de $T_{\mathbf{L}}$-copula wordt gebruikt. De vierde sectie bundelt opnieuw de resultaten en de laatste sectie bevat dan de bewijzen van deze resultaten. Het blijkt dat, alhoewel de definities van de spelvarianten enkel verschillen in de gebruikte copula, de karakterisatie van de optimale strategieën toch volledig verschilt voor elke spelvariant.

In Hoofdstuk 8 worden standaard n-dupletten en standaard n-tripletten, die bepaalde collecties van verzamelingen van strikt positieve getallen behelzen, gepartitioneerd door gebruik te maken van hun zogenaamd straatnummer, met als doel te bepalen hoeveel van die collecties een gegeven straatnummer hebben. De eerste sectie introduceert het straatnummer en toont het verband met de opbrengstmatrix van de in Hoofdstuk 5 gedefinieerde spellen aan. De tweede sectie bepaalt dan, voor een gegeven straatnummer, hoeveel n-dupletten dit straatnummer hebben. Het blijkt dat hiervoor bepaalde concepten uit de partitietheorie nodig zijn. De derde sectie introduceert dan de zogenaamde duale partitieset, wat leidt tot een interessante methode om trapdriehoeken te maken met behulp van rechthoeken. De vierde en tevens laatste sectie behandelt dan het straatnummer van n-tripletten. Er wordt aangetoond dat het zeer moeilijk wordt om algemene resultaten te geven betreffende het aantal n-tripletten die een bepaald straatnummer hebben.

Bibliography

[1] D. Alonso and P. Fernández-Berrocal. Irrational decisions: attending to numbers rather than ratios. *Personality and Individual Differences*, 35:1537–1547, 2003.

[2] C. Alsina, R. Nelsen, and B. Schweizer. On the characterization of a class of binary operations on distribution functions. *Statistics & Probability Letters*, 17:85–89, 1993.

[3] G.E. Andrews. *The theory of partitions*. Cambridge University Press, 1984.

[4] K. Basu. Fuzzy revealed preference theory. *Journal of Economic Theory*, 32:212–227, 1984.

[5] V. Bawa, E. Lindenberg, and L. Rafsky. An efficient algorithm for determining stochastic dominance admissible sets. *Management Science*, 25:609–622, 1979.

[6] M.H. Bazerman, H. Schroth, P.P. Shah, K.A. Diekmann, and A.E. Tenbrunsel. The inconsistent role of comparison others and procedural justice in reactions to hypothetical job descriptions: Implications for job acceptance decisions. *Organizational Behavior and Human Decision Processes*, 60:326–352, 1994.

[7] M.H. Bazerman, A.E. Tenbrunsel, and K.A. Wade-Benzoni. Negotiating with yourself and losing: Understanding and managing competing internal preferences. *Academy of Management Review*, 23:225–241, 1998.

[8] N.H. Bazerman, D.A. Moore, A.E. Tenbrunsel, K.A. Wade-Benzoni, and S. Blount. Explaining how preferences change across joint versus separate evaluation. *Journal of Economic Behavior & Organization*, 39:41–58, 1999.

[9] U. Bodenhofer. Representations and constructions of similarity-based fuzzy orderings. *Fuzzy Sets and Systems*, 137:113–136, 2003.

[10] R. Brualdi and G. Hwang. Generalized transitive tournaments and doubly stochastic matrices. *Linear Algebra and its Applications*, 172:151–168, 1992.

[11] F. Chiclana, F. Herrera, and E. Herrera-Viedma. Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations. *Fuzzy Sets and Systems*, 97:33–48, 1998.

[12] R.P. Cubbit, A. Munro, and C. Starmer. Testing explanations of preference reversal. *The Economic Journal*, 114:709–726, 2004.

[13] M. Dasgupta and R. Deb. Fuzzy choice functions. *Social Choice and Welfare*, 8:171–182, 1991.

[14] M. Dasgupta and R. Deb. Transitivity and fuzzy preferences. *Social Choice and Welfare*, 13:305–318, 1996.

[15] H. David. *The Method of Paired Comparisons*, volume 12. Griffin's Statistical Monographs & Courses, 1963.

[16] B. De Baets, H. De Meyer, and B. De Schuymer. Cycle-transitivity of reciprocal relations. In *Principles of Fuzzy Preference Modelling and Decision Making*, pages 165–181, 2003.

[17] B. De Baets, H. De Meyer, and B. De Schuymer. Self-dual types of cycle-transitivity. In *EUSFLAT*, pages 292–295, 2003.

[18] B. De Baets, H. De Meyer, and B. De Schuymer. *Logical, Algebraic, and Probabilistic Aspects of Triangular Norms*, chapter Transitive Comparison of Random Variables. Elsevier, 2005.

[19] B. De Baets, H. De Meyer, B. De Schuymer, and S. Jenei. Cyclic evaluation of transitivity of reciprocal relations. *Social Choice and Welfare*, 2005 (in press).

[20] B. De Baets and J. Fodor. Twenty years of fuzzy preference structures (1978–1997). *Rivista di Matematica per le Scienze Economiche e Sociali*, 20:45–66, 1997.

[21] B. De Baets, B. Van de Walle, and E. Kerre. Fuzzy preference structures without incomparability. *Fuzzy Sets and Systems*, 76:333–348, 1995.

[22] H. De Meyer, B. De Baets, and B. De Schuymer. Transitive comparison of marginal probability distributions. In *EUSFLAT*, pages 296–299, 2003.

[23] H. De Meyer, B. De Baets, and B. De Schuymer. Transitive comparison of random variables. In *Principles of Fuzzy Preference Modelling and Decision Making*, pages 249–265, 2003.

[24] H. De Meyer, B. De Baets, and B. De Schuymer. On the transitivity of the comonotonic and countermonotonic comparison of random variables. *Journal of Multivariate Analysis*, 2005 (accepted).

[25] B. De Schuymer, H. De Meyer, and B. De Baets. A fuzzy approach to stochastic dominance of random variables. In *IFSA*, pages 253–260, 2003.

[26] B. De Schuymer, H. De Meyer, and B. De Baets. Playing games with dice and ordered lists. In *Principles of Fuzzy Preference Modelling and Decision Making*, pages 215–228, 2003.

[27] B. De Schuymer, H. De Meyer, and B. De Baets. Optimal strategies of some symmetric matrix games. In *FSSCEF*, pages 162–169, 2004.

[28] B. De Schuymer, H. De Meyer, and B. De Baets. Cycle-transitive comparison of independent random variables. *Journal of Multivariate Analysis*, 2005 (in press).

[29] B. De Schuymer, H. De Meyer, and B. De Baets. Extreme copulas and the comparison of ordered lists. *Theory and Decision*, 2005 (submitted).

[30] B. De Schuymer, H. De Meyer, and B. De Baets. Optimal strategies for equal-sum dice games. *Discrete Optimization*, 2005 (submitted).

[31] B. De Schuymer, H. De Meyer, and B. De Baets. Optimal strategies for symmetric matrix games with partitions. *Discrete Applied Mathematics*, 2005 (submitted).

[32] B. De Schuymer, H. De Meyer, B. De Baets, and S. Jenei. On the cycle-transitivity of the dice model. *Theory and Decision*, 54:261–285, 2003.

[33] B. De Schuymer, H. De Meyer, S. Jenei, and B. De Baets. Transitive properties of reciprocal fuzzy relations generated by a multiset utility model. In *EUROFUSE*, pages 219–224, 2002.

[34] V. Denes-Raj and S. Epstein. Conflict between intuitive and rational processing: When people behave against their better judgment. *Journal of Personality and Social Psychology*, 66:819–829, 1994.

[35] J.P. Doignon, B. Monjardet, M. Roubens, and P. Vincke. Biorder families, valued relations and preference modelling. *Journal of Mathematical Psychology*, 30:435–480, 1986.

[36] B. Dutta and J.F. Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16:513–532, 1999.

[37] H. Falk and H. Levy. Market reaction to quarterly earnings' annoncements: a stochastic dominance based test of market efficiency. *Management Science*, 35:425–446, 1989.

[38] P. Fishburn. Binary choice probabilities: On the varieties of stochastic transitivity. *Journal of Mathematical Psychology*, 10:327–352, 1973.

[39] J. Fodor and T. Keresztfalvi. A characterization of the hamacher family of t-norms. *Fuzzy Sets and System*, 65:51–58, 1994.

[40] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[41] M. Frank. On the simultaneous associativity of $f(x,y)$ and $x+y-f(x,y)$. *Aequationes Mathematicae*, 17:194–226, 1979.

[42] J. García-Lapresta and B. Llamazares. Aggregation of fuzzy preferences: some rules of the mean. *Social Choice and Welfare*, 17:673–690, 2000.

[43] J García-Lapresta and L.C. Meneses. An empirical analysis of transitivity with four scaled preferential judgment modalities. *Review of Economic Design*, 8:335–346, 2003.

[44] J. García-Lapresta and B. Llamazares. Majority decisions based on difference of votes. *Journal of Mathematical Economics*, 35:463–481, 2001.

[45] S.I. Gass. Tournaments, transitivity and pairwise comparison matrices. *Journal of the Operational Research Society*, 49:616–624, 1998.

[46] C. Genest, J. Molina, J. Lallena, and C. Sempi. A characterization of quasi-copulas. *Journal of Multivariate Analysis*, 69:193–205, 1999.

[47] D.M. Grether and C.R. Plott. Economic theory of choice and the preference reversal phenomenon. *The American Economic Review*, 69:623–638, 1979.

[48] H. Hamacher. Über logische Verknüpfungen unscharfer Aussagen und deren zugehörige Bewertungsfunktionen. *Working Paper 75/14, RWTH Aachen*, 1975.

[49] C.K. Hsee. The evaluability hypothesis: An explanation for preference reversals between joint and separate evaluations of alternatives. *Organizational Behavior and Human Decision Processes*, 67:247–257, 1996.

[50] C.K. Hsee, S. Blount, G.F. Loewenstein, and M.H. Bazerman. Preference reversals between joint and separate evaluations of options: a review and theoretical analysis. *Psychological Bulletin*, 125:576–590, 1999.

[51] J.R. Irwin, P. Slovic, S. Lichtenstein, and G.H. McClelland. Preference reversals and the measurement of environmental values. *Journal of Risk and Uncertainty*, 6:5–18, 1993.

[52] H. Joe. *Multivariate Models and Dependence Concepts*. Chapman & Hall/CRC, New York, 1997.

[53] J. Kacprzyk, M. Fedrizzi, and H. Nurmi. Group decision making and consensus under fuzzy preferences and fuzzy majority. *Fuzzy Sets and Systems*, 49:21–31, 1992.

[54] J. Kacprzyk, H. Nurmi, and M. Fedrizzi, editors. *Consensus under Fuzziness*. Kluwer Academic Publishers, Boston, 1996.

[55] M.G. Kendall and B. Babington Smith. On the method of paired comparison. *Biometrika*, 31:324–345, 1940.

[56] E. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic, Studia Logica Library*. Kluwer Academic Publishers, 2000.

[57] E. Klement, R. Mesiar, and E. Pap. Invariant copulas. *Kybernetika*, 38:275–285, 2002.

[58] S. Lahiri. Axiomatic characterizations of threshold choice functions for comparison functions. *Fuzzy Sets and Systems*, 132:77–82, 2002.

[59] H. Levy. *Stochastic Dominance*. Kluwer Academic Publishers, Norwell, MA, 1998.

[60] R. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–193, 1973.

[61] P. Mikusiński, H. Sherwood, and M.D. Taylor. The Fréchet bounds revisited. *Real Analysis Exchange*, 17:759–764, 1991–92.

[62] F. Milne and E. Neave. Dominance relations among standardized variables. *Management Science*, 40:1343–1352, 1994.

[63] B. Monjardet. *Lecture Notes in Economics and Mathematical Systems*, volume 301, chapter A generalisation of probabilistic consistency: linearity conditions for valued preference relations, pages 36–53. Springer-Verlag, Berlin, 1988.

[64] R. Nelsen. *An Introduction to Copulas*. Springer, New York, 1998.

[65] H. Nurmi. *Comparing Voting Systems*. Reidel, Dordrecht, 1987.

[66] I. Ritov and D. Kahneman. *Environment, Ethics, and Behavior*, chapter How people value the environment? New Lexington Press, San Francisco, CA, 1997.

[67] F. Roberts. Homogeneous families of semiorders and the theory of probabilistic consistency. *Journal of Mathematical Psychology*, 8:248–263, 1971.

[68] S. Saminger, R. Mesiar, and U. Bodenhofer. Domination of aggregation operators and preservation of transitivity. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10:11–35, 2002.

[69] B. Schweizer and A. Sklar. Operations on distribution functions not derivable from operations on random variables. *Studia Mathematica*, 52:43–52, 1974.

[70] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, 1983.

[71] A. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231, 1959.

[72] C.E. Sumpter, W. Temple, and T.M. Foster. The transitivity of choices between different response requirements. *Journal of the Experimental Analysis of Behavior*, 72:235–249, 1999.

[73] S. Switalski. Rationality of fuzzy reciprocal preference relations. *Fuzzy Sets and Systems*, 107:187–190, 1999.

[74] S. Switalski. Transitivity of fuzzy preference relations – an empirical study. *Fuzzy Sets and Systems*, 118:503–508, 2001.

[75] S. Switalski. General transitivity conditions for fuzzy reciprocal preference matrices. *Fuzzy Sets and Systems*, 137:85–100, 2003.

[76] S. Switalski. Half-integrality of vertices of the generalized transitive tournament polytope ($n = 6$). *Discrete Mathematics*, 271:251–260, 2003.

[77] T. Tanino. Fuzzy preference orderings in group decision making. *Fuzzy Sets and Systems*, 12:117–131, 1984.

[78] T. Tanino. *Non-Conventional Preference Relations in Decision Making*, volume 301 of *Lecture Notes in Economics and Mathematical Systems*, chapter Fuzzy preference relations in group decision making, pages 54–71. Springer-Verlag, Berlin, 1988.

[79] B. Van de Walle, B. De Baets, and E. Kerre. Characterizable fuzzy preference structures. *Annals of Operations Research, Special Issue "Preference modelling"*, 80:105–136, 1998.

[80] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

[81] N.N. Vorob'eb. *Game Theory: Lectures for Economists and Systems Scientists*. Springer-Verlag, New York, 1977.

[82] N.N. Vorob'eb. *Foundations of Game Theory: Noncooperative Games*. Birkhauser Verlag, Boston, 1994.

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